

Bifurcating nodoids

Rafe Mazzeo*
Stanford University

Frank Pacard†
Université Paris XII

November 1, 2001

Abstract

Complete, axially symmetric surfaces of constant mean curvature in \mathbb{R}^3 necessarily lie in a one-parameter family of so-called Delaunay surfaces, denoted D_τ . The embedded elements of this family, corresponding to $\tau \in (0, 1]$ and known as unduloids, are stable in the sense that the only global constant mean curvature deformations of them are other elements of this Delaunay family. We prove here that this same property is true for the immersed Delaunay surfaces parametrized by $\tau < 0$ and known as nodoids, only when τ is sufficiently close to zero (this corresponds to the necksize of these surfaces being small). On the other hand, as τ decreases to $-\infty$, we prove the existence of infinitely many branches of complete, cylindrically bounded constant mean curvature surfaces which bifurcate off of this Delaunay family. These branches contain surfaces which have only a discrete symmetry group.

1 Introduction

In 1841, C. Delaunay discovered a beautiful one-parameter family of complete noncompact surfaces of constant mean curvature one in \mathbb{R}^3 which are invariant under rotations about an axis [2]. (Henceforth we shall abbreviate constant mean curvature one by CMC.) Using the rotational symmetry, the search for these surfaces reduces to finding their meridian curves, and these may be found, in turn, by solving an appropriate ODE. This family of ‘Delaunay surfaces’ is parametrized by a variable τ in the set

$$\mathcal{P} = (-\infty, 1] - \{0\} = \mathcal{P}^+ \cup \mathcal{P}^-; \quad \mathcal{P}^- = (-\infty, 0), \quad \mathcal{P}^+ = (0, 1].$$

The Delaunay surface corresponding to any value $\tau \in \mathcal{P}$ will be denoted D_τ . When $\tau \in \mathcal{P}^+$, then D_τ is embedded and is called an unduloid; the meridian curve in this case is the roulette of an ellipse. When $\tau \in \mathcal{P}^-$, then D_τ is no longer embedded, and is called a nodoid; the meridian curve of any nodoid is the roulette of a hyperbola. There is a good geometric limit of the surfaces D_τ as $\tau \rightarrow 0$ (which is the same for $\tau \searrow 0$ and $\tau \nearrow 0$), which is the noded surface formed by the infinite union of mutually tangent spheres of radius 1 arranged along a common axis. A nice geometric description of these Delaunay surfaces can be found in [3].

In this paper we shall prove the existence of some rather surprising new families of complete CMC surfaces which arise as deformations of Delaunay nodoids. Let us digress briefly to set these new surfaces into their proper context before describing our results in more detail.

The past several years have witnessed great progress in the understanding of more general complete, CMC surfaces of finite topology. On the one hand, the great variety and flexibility of such surfaces has been suggested by numerous numerical and computer experiments, notably by the GANG group at the University of Massachusetts and Grosse-Brauckmann and his collaborators at the Universität Bonn, see <http://www.gang.umass.edu/cmc/> for many nice pictures.

*Supported by the NSF under grant DMS-9971975 and at MSRI by NSF grant DMS-9701755

†Supported at MSRI by NSF grant DMS-9701755

On the other hand, there have been several theoretical advances which vindicate many of these computer experiments. These advances may be subdivided into two groups of results, those concerning various constructions of complete CMC surfaces, which are not relevant to the present discussion, and those concerning the structure theory of these surfaces.

This structure theory is much better developed when we restrict attention to complete (oriented) Alexandrov-embedded CMC surfaces of finite topology, the key point being that any one of these surfaces has only finitely many ends, and each of these ends is necessarily modeled on a Delaunay unduloid. Recall first that any surface of this type is conformally equivalent to a punctured Riemann surface $\Sigma = \bar{\Sigma} \setminus \{p_1, \dots, p_k\}$; let Y^3 be a handlebody such that $\partial Y = \bar{\Sigma}$. Then Σ is said to be Alexandrov embedded if its immersion into \mathbb{R}^3 extends to an immersion of $Y \setminus \{p_1, \dots, p_k\}$. In particular, D_τ is not Alexandrov-embedded when $\tau < 0$.

A deep theorem of Meeks [12] states that any end of an Alexandrov-embedded CMC surface of finite topology is cylindrically bounded. Following this, Korevaar, Kusner and Solomon [5] showed that for any such end there is a Delaunay unduloid D_τ to which the end converges exponentially. [4] contains a further global structure theorems. These results show that Delaunay unduloids play a fundamental rôle as building blocks for finite topology Alexandrov-embedded CMC surfaces. In addition, this tameness of the geometry of the ends leads to a fairly detailed understanding of many analytic and geometric problems on these surfaces, [7], [10].

It is natural to try to extend this theory to complete, CMC surfaces with finite topology which are not necessarily Alexandrov-embedded, for example those with ends modeled on Delaunay nodoids. Fairly general examples of such surfaces are constructed in [9], [10], but the ends of those surfaces are modeled on Delaunay nodoids with τ very close to zero.

The results of this paper begin to elucidate the limitations of the structure theory [5], and how it fails when Alexandrov-embeddedness is dropped. More specifically, we examine the stability theory of the Delaunay nodoids D_τ as $\tau \rightarrow -\infty$. In a sense we explain below, there is some stable range $-\tau_* < \tau < 0$ for which the only deformations of D_τ are to other Delaunay surfaces. However, as τ decreases, we establish the existence of new CMC surfaces which bifurcate from the Delaunay family. These surfaces have only a discrete symmetry group, rather than full rotational symmetry, and the number and geometric complexity of these bifurcations increases as τ becomes more negative. All of these new surfaces remain cylindrically bounded, but their geometric structure is rather intricate.

To conclude this general discussion, we recall that the basic theory of the moduli space of complete Alexandrov-embedded CMC surfaces of finite topology is established in [7]. The paper [10] extends this theory to include CMC surfaces with ends modeled on Delaunay nodoids with $\tau > \tau_*$; (it also develops some global aspects of this moduli space theory). Our results here show that any complete description of the moduli space theory of complete CMC surfaces of finite topology must incorporate these new families of CMC surfaces.

We may now give the precise statements of our main results. These require a few preliminary definitions.

Definition 1 *We set terminology for a family of rigid motions in \mathbb{R}^3 , normalized for convenience to fix the z -axis, as well as the surfaces which have the associated symmetries.*

- i) *Let $j \in \mathbb{N}$, $j \geq 2$. Let R_j denote the rotation by angle $2\pi/j$ about the z -axis in \mathbb{R}^3 . We say that a surface $\Sigma \subset \mathbb{R}^3$ is R_j -symmetric if it is invariant by R_j , but not by any $R_{j'}$ for $j' > j$.*
- ii) *For $\alpha \in [-\pi, \pi]$, let R_α denote the rotation by angle α in \mathbb{R}^2 . Also, let $t_\alpha \in \mathbb{R}$. We say that the rigid motion S_α of \mathbb{R}^3 is a screw motion about the z -axis (of angle α and translation length t_α) if it has the form*

$$S_\alpha : \begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R} & \longrightarrow & \mathbb{R}^2 \times \mathbb{R} \\ (x_1, x_2, z) & \longmapsto & (R_\alpha x, z + t_\alpha) \end{array}$$

We say that the surface $\Sigma \subset \mathbb{R}^3$ is S_α -symmetric if it is invariant with respect to some screw motion S_α , for some choice of translation length t_α , and if it is not invariant with respect to any other screw motion $S_{\alpha'}$ with $0 < \alpha' < \alpha$ if $\alpha \in (0, \pi]$ or $\alpha < \alpha' < 0$ if $\alpha \in [-\pi, 0)$.

iii) Finally, we say that the surface $\Sigma \subset \mathbb{R}^3$ is $T_{j,\alpha}$ -symmetric if it is both R_j -symmetric and S_α -symmetric.

Our first main result states that there are infinitely many bifurcations from the Delaunay family D_τ as $\tau \rightarrow -\infty$ to new CMC surfaces which have a discrete symmetry group:

Theorem 1 *Let $j \geq 2$; then there for any $\alpha \in [-\pi/j, \pi/j] - \{0\}$, there exists a $T_{j,\alpha}$ -symmetric CMC surface. In fact, each of these surfaces lies in a branch which bifurcates from the nodoid D_{τ_j} at some value $\tau_j < -\sqrt{j^2 - 2}$.*

This may be proved either using the general bifurcation theorem of Smoller and Wasserman [15] or more simply by degree theory [14]. As in most such problems, the main point is to show that the index (i.e. number of negative eigenvalues) of the Jacobi operator L_τ on some compact quotient of D_τ increases as $\tau \searrow -\infty$. As part of this, we show that there exists some value $\tau_* \in (-2, -\sqrt{2})$ such that no bifurcations occur when $\tau_* < \tau < 0$.

This result can be sharpened when j is large enough, for in this case we can guarantee that the bifurcation value τ_j is large negative. This allows us to obtain better control on the spectrum of Jacobi operator, and so we may use the bifurcation theorem of Crandall and Rabinowitz [1] to show that the bifurcation branches are smooth.

Theorem 2 *There exists $j_0 \geq 2$ such that when $j \geq j_0$ and $\alpha \in [-\pi/j, \pi/j] - \{0\}$, then there are values $\tau_{j,\alpha} < 0$ $\eta_{j,\alpha} > 0$, and a real analytic branch of $T_{j,\alpha}$ -symmetric CMC surfaces, which we denote $D_{j,\alpha}(\tau, \eta)$, parametrized by η with $|\eta| \leq \eta_{j,\alpha}$, which bifurcate from $D_{\tau_{j,\alpha}}$. These families of surfaces have the following properties:*

- (i) *As $\eta \rightarrow 0$, the surfaces $D_{j,\alpha}(\eta)$ converge uniformly on compact subsets to the nodoid $D_{\tau_{j,\alpha}}$, in C^∞ topology.*
- (ii) *Locally on $D_{\tau_{j,\alpha}}$ and for η sufficiently small, we can write $D_{j,\alpha}(\eta)$ as a normal graph over $D_{\tau_{j,\alpha}}$ for some function w_η . The functions w_η/η converge uniformly on compact sets to a nontrivial function of the form $(s, \theta) \rightarrow \phi(s) \cos(j\theta)$. (The variables (s, θ) will be defined in §2.) Furthermore, the function $\psi(s, \theta) := \phi(s) e^{i\theta}$ is a nontrivial solution of $\mathcal{L}_\tau \psi = 0$ which satisfies $\phi(s + 2\pi s_\tau, \theta) = e^{ij\alpha} \phi(s, \theta)$.*

In fact, we prove that

$$\tau_{j,\alpha} = -\frac{j}{\sqrt{1 - (\alpha j/2\pi)^2}} + \mathcal{O}(j^{-1}). \quad (1)$$

Remark 1 *It is likely that Theorem 2 holds whenever $j \geq 2$. We also suspect that for any $j \geq 2$, $|\alpha| \leq \pi/j$, $\alpha \neq 0$, there are precisely two bifurcating branches, and these correspond to two separate bands of continuous spectrum of \mathcal{L}_τ , with one branch bifurcating at the value $\tau = \tau_{j,\alpha}$ above, which lies in the first band, and the other bifurcating at a value $\bar{\tau}_{j,\tau}$ lying in the second band. This second bifurcation point should satisfy*

$$\bar{\tau}_{j,\alpha} = -\frac{j}{\sqrt{1 - (|\alpha j|/2\pi - 1)^2}} + \mathcal{O}(j^{-1}).$$

as $j \rightarrow \infty$.

2 Isothermal parametrization of Delaunay surfaces

Since the Delaunay surfaces D_τ are surfaces of revolution, their most natural parametrizations would seem to be the obvious ones:

$$(u, \theta) \longmapsto (\rho(u) \cos \theta, \rho(u) \sin \theta, u),$$

where function $\rho = \rho_\tau$ is a solution of an ODE which is derived from the constant mean curvature condition. However, for most analytic purposes it turns out to be far more convenient to use a different, isothermal parametrization, which we now describe. This definition may seem ad hoc, but is motivated by a systematic line of reasoning in the theory of integrable systems; [9] contains a detailed derivation of the fact that this is parametrization of the D_τ .

This isothermal parametrization rests on two functions $\sigma = \sigma_\tau$ and $\kappa = \kappa_\tau$, the definitions of which vary, according to whether τ is positive or negative. First, σ is the unique smooth nonconstant solution to the initial value problem

$$\left(\frac{d\sigma}{ds}\right)^2 + \tau^2 \cosh^2 \sigma = 1, \quad \partial_s \sigma(0) = 0, \quad \sigma(0) < 0, \quad \text{when } \tau \in \mathcal{P}^+, \quad (2)$$

$$\left(\frac{d\sigma}{ds}\right)^2 + \tau^2 \sinh^2 \sigma = 1, \quad \partial_s \sigma(0) = 0, \quad \sigma(0) < 0, \quad \text{when } \tau \in \mathcal{P}^-. \quad (3)$$

Next, $\kappa(s)$ is the unique solution of

$$\frac{d\kappa}{ds} = \tau^2 e^\sigma \cosh \sigma, \quad \kappa(0) = 0, \quad \text{when } \tau \in \mathcal{P}^+, \quad (4)$$

$$\frac{d\kappa}{ds} = -\tau^2 e^\sigma \sinh \sigma, \quad \kappa(0) = 0, \quad \text{when } \tau \in \mathcal{P}^-. \quad (5)$$

The change of variables from the previous cylindrical coordinates (u, θ) is effected by setting $u = \kappa(s)$ and one then has (see [9]) $\rho(\kappa(s)) = \tau e^{\sigma(s)}$, and so the new isothermal parametrization is given by

$$X_\tau : \mathbb{R} \times S^1 \ni (s, \theta) \longmapsto \frac{1}{2} \left(\tau e^{\sigma_\tau(s)} \cos \theta, \tau e^{\sigma_\tau(s)} \sin \theta, \kappa_\tau(s) \right). \quad (6)$$

The metric coefficients in this new coordinate system are $g_{ss} = g_{\theta\theta} = \tau^2 e^{2\sigma}$, $g_{s\theta} = g_{\theta s} = 0$.

Regardless of the sign of τ , the function $s \rightarrow \sigma_\tau(s)$ necessarily changes sign. Together with (5), this means that when $\tau < 0$, then κ is not monotone; in contrast, using (4), κ is monotone when $\tau > 0$. Hence although the differential of X_τ is always full rank, so that X_τ is always an immersion, it is only an embedding when $\tau > 0$. Notice also that when $\tau = 1$, then (2) implies that $\sigma \equiv 0$, and so $\kappa(s) = s$ and $\rho \equiv 1/2$, which means that D_1 is a cylinder of radius $1/2$.

3 The period function

The Hamiltonian nature of the equations (2) and (3) implies that σ is periodic. We denote its period by a $2\pi s_\tau$. We now investigate the dependence of s_τ on τ .

To begin, s_τ has an integral representation. To state it, let $a_\tau, a'_\tau > 0$ be determined by the equations $\tau \cosh a_\tau = 1$ when $\tau > 0$ and $\tau \sinh a'_\tau = -1$ when $\tau < 0$. Then

$$s_\tau := \frac{2}{\pi} \int_0^{a_\tau} \frac{d\sigma}{\sqrt{1 - \tau^2 \cosh^2 \sigma}}, \quad \text{when } \tau \in \mathcal{P}^+, \quad (7)$$

and

$$s_\tau := \frac{2}{\pi} \int_0^{a'_\tau} \frac{d\sigma}{\sqrt{1 - \tau^2 \sinh^2 \sigma}}, \quad \text{when } \tau \in \mathcal{P}^-. \quad (8)$$

We also use the following equivalent forms of these representations

$$s_\tau := \frac{2}{\pi} \int_0^{b_\tau} \frac{dx}{\sqrt{\cos^2 x - \tau^2}}, \quad \text{when } \tau \in \mathcal{P}^+, \quad (9)$$

where $b_\tau = \arccos \tau \in [0, \pi/2)$, $0 < \tau \leq 1$, and

$$s_\tau := \frac{2}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{\cos^2 x + \tau^2}}, \quad \text{when } \tau \in \mathcal{P}^-. \quad (10)$$

These are effected by the changes of variables $\tau \cosh \sigma = \cos x$ and $\tau \sinh \sigma = \cos x$, respectively.

Proposition 1 *As a function of τ , s_τ is monotone increasing when $-\infty < \tau < 0$ and monotone decreasing when $0 < \tau \leq 1$.*

Proof: It is obvious from (10) that $ds_\tau/d\tau > 0$ when $\tau < 0$. The fact that s_τ decreases when τ increases from 0 to 1 is less obvious and somewhat more difficult to obtain; since we do not need it here we omit the proof. \square

We next consider the asymptotics of s_τ as τ approaches various possible limiting values in \mathcal{P} . The first two cases are less important, but we record them anyway; namely, we have

$$s_\tau = -\frac{1}{\pi} \log \tau^2 + \mathcal{O}(1) \quad \text{as } \tau \rightarrow 0 \quad (11)$$

and also

$$\lim_{\tau \nearrow 1} s_\tau = 1.$$

On the other hand, the behaviour of s_τ as $\tau \rightarrow -\infty$ is fundamental to our analysis. To study this we introduce an auxiliary function

$$\gamma_\tau(t) := \tau \sigma_\tau(s_\tau t), \quad \tau < 0;$$

by design, γ has period 2π .

Lemma 1 *As $\tau \rightarrow -\infty$, γ_τ converges uniformly to the function $\cos t$. More precisely,*

$$\gamma_\tau(t) = \cos t + \mathcal{O}(|\tau|^{-2}) \quad \text{and} \quad \partial_\tau \gamma_\tau(t) = \mathcal{O}(|\tau|^{-3}), \quad (12)$$

as $\tau \rightarrow -\infty$. Simultaneously, in the same limit

$$s_\tau = -\frac{1}{\tau} + \mathcal{O}(|\tau|^{-3}), \quad \text{and} \quad \partial_\tau s_\tau = \frac{1}{\tau^2} + \mathcal{O}(|\tau|^{-4}). \quad (13)$$

Remark 2 *The proof below easily extends to give full asymptotic expansions for s_τ and γ in powers of $1/\tau$ as $\tau \rightarrow -\infty$. However, we only require the terms given in this statement.*

Proof: Set $\sigma_0 := \sigma(0)$. Since $|\sigma| \leq -\sigma_0$, we may define the function $w(s)$ by

$$\sigma(s) = \sigma(0) \cos(\tau w(s)).$$

To normalize it, we require that $w(0) = 0$ and $w'(0) > 0$. The equation for σ , (3), becomes

$$(\partial_s w)^2 + \frac{\sinh^2 \sigma - \sinh^2 \sigma_0}{\sigma^2 - \sigma_0^2} = 0,$$

which can be rewritten as

$$(\partial_s w)^2 = \frac{\Phi(\sigma^2) - \Phi(\sigma_0^2)}{\sigma^2 - \sigma_0^2} \quad \text{with} \quad \Phi(t) := \sinh^2 \sqrt{t}. \quad (14)$$

(3) also implies that $|\sigma| \leq |\sinh \sigma| \leq |\tau|^{-1}$. Using the first term of the Taylor expansion of Φ gives $\partial_s w = 1 + \mathcal{O}(|\tau|^{-2})$, and so we have

$$w(s) = s(1 + \mathcal{O}(|\tau|^{-2})). \quad (15)$$

Next, the periodicity of σ translates to the equality

$$-2\pi = \tau w(2\pi s_\tau) = \tau(2\pi s_\tau + \mathcal{O}(s_\tau |\tau|^{-2})),$$

or equivalently,

$$s_\tau = -\frac{1}{\tau} + \mathcal{O}(|\tau|^{-3}).$$

Inserting this into the expression for $\gamma(t)$, and using $\sigma_0 = \operatorname{arcsinh}(1/\tau) = 1/\tau + \mathcal{O}(1/|\tau|^3)$ gives

$$\gamma(t) = \cos t + \mathcal{O}(|\tau|^{-2}).$$

This procedure may be continued, using this expansion for γ (and hence for σ) to determine the next term in the expansion for s_τ , then using this in turn to get the next term in the expansion for γ , and so on. \square

4 Spectral analysis of the Jacobi operator

The primary tool in this paper is a detailed analysis of the spectrum of the linearized mean curvature operator \mathcal{L}_τ (which is usually called the Jacobi operator) on the family of Delaunay surfaces D_τ . We begin by describing the specific form of this operator and then review how Bloch wave theory may be applied in this context. This theory reduces the spectral analysis to that of a countable collection of continuous families of operators with discrete spectrum and it shows that the spectrum of \mathcal{L}_τ is a union of ‘bands’. Geometric considerations allow us to identify specific solutions of $\mathcal{L}_\tau u = 0$, and this, in turn, allows us to track the location of some of these bands as $\tau \rightarrow -\infty$.

4.1 Coordinate expression

It is well known that the Jacobi operator has the form

$$\mathcal{L}_\tau := -\Delta_\tau - |A_\tau|^2,$$

where Δ_τ and A_τ are the Laplace operator and second fundamental form on D_τ , respectively. In terms of the isothermal parametrization (6),

$$\mathcal{L}_\tau = -\frac{1}{\tau^2 e^{2\sigma}} (\partial_s^2 + \partial_\theta^2 - \tau^2 \cosh 2\sigma). \quad (16)$$

We shall be using the periodicity of the coefficients of this operator in a central way. However, it is more convenient to work with a family of operators for which the period is fixed as τ varies. Accordingly, we change variables, setting

$$s := s_\tau t,$$

so that the coefficients of L_τ with respect to (t, θ) have period 2π for all $\tau \in \mathcal{P}$. In these new coordinates,

$$\mathcal{L}_\tau = -\frac{1}{s_\tau^2 \tau^2 e^{2\sigma}} (\partial_t^2 + s_\tau^2 \partial_\theta^2 - s_\tau^2 \tau^2 \cosh(2\sigma)).$$

Removing the factor $1/(s_\tau^2 \tau^2 e^{2\sigma})$, we define

$$L_\tau := -\partial_t^2 - s_\tau^2 \partial_\theta^2 - s_\tau^2 \tau^2 \cosh(2\sigma).$$

Since we are only interested in some aspects of the spectral analysis of \mathcal{L}_τ , such as the number of negative eigenvalues, the existence of a nullspace, etc., we may concentrate on the study of the slightly simpler operator L_τ .

4.2 Spectral decomposition

The operator L_τ has many symmetries, and these may be used to reduce it to operators for which the spectral analysis is more tractable.

The first and most obvious reduction uses the rotational invariance in θ . Thus, if $u(t, \theta) \in L^2(\mathbb{R} \times S^1)$, we have the decomposition

$$u(t, \theta) = \sum_{j \in \mathbb{Z}} u_j(t) e^{ij\theta},$$

where each of the coefficients $u_j(t)$ is in $L^2(\mathbb{R})$ and $\sum \|u_j\|^2 < \infty$. The operator L_τ induces the operator

$$L_{\tau,j} = -\partial_t^2 + s_\tau^2 j^2 - s_\tau^2 \tau^2 \cosh(2\sigma), \quad (17)$$

on the j^{th} eigenspace. In fancier language, the operator L_τ is reduced by the splitting

$$L^2(\mathbb{R} \times S^1) = \bigoplus_{j \in \mathbb{Z}} L_j^2(\mathbb{R})$$

from this eigenspace decomposition into a direct sum of (self adjoint) ordinary differential operators $L_{\tau,j}$. Note that $L_{\tau,j} = L_{\tau,-j}$, even though the spaces L_j^2 and L_{-j}^2 are different (one corresponds to the eigenfunction $e^{ij\theta}$ and the other to the eigenfunction $e^{-ij\theta}$). It is clear that

$$\text{spec}(L_\tau) = \bigcup_{j \in \mathbb{N}} \text{spec}(L_{\tau,j}).$$

Furthermore, noting that

$$L_{\tau,j} = L_{\tau,0} + s_\tau^2 j^2, \quad (18)$$

we may as well restrict attention to the operator $L_{\tau,0}$ and the single Hilbert space $L^2(\mathbb{R})$.

To analyze this last operator we use the technique of Bloch waves, also known as Floquet theory. We refer to [13] and [8] for details, cf. also [11]. This relies on a direct integral decomposition of $L^2(\mathbb{R})$ defined using the Fourier-Laplace transform. Given any function f in the Schwartz space \mathcal{S} on \mathbb{R} , define

$$\hat{f}(t, \alpha) = \sum_{j \in \mathbb{Z}} f(t + 2\pi j) e^{-i\alpha j}.$$

We may as well assume that $-\pi \leq \alpha \leq \pi$. This function satisfies $\hat{f}(t + 2\pi, \alpha) = e^{i\alpha} \hat{f}(t, \alpha)$, and so we lose no information by restricting t to lie in $[-\pi, \pi]$. It is straightforward to check that

$$\|f\|_{L^2(\mathbb{R})}^2 = 2\pi \|\hat{f}\|_{L^2([-\pi, \pi]^2)}^2,$$

which means that the map $f \mapsto \hat{f}$ may be extended as an isometry from $L^2(\mathbb{R})$ into $L^2([-\pi, \pi]^2)$. In fact it is an isometry, since there is an inversion mapping given by

$$f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\bar{t}, \alpha) e^{i\alpha j} d\alpha,$$

where $\bar{t} \in [-\pi, \pi]$ and $t = \bar{t} + 2\pi j$. Altogether, this gives the direct integral decomposition

$$L^2(\mathbb{R}) = \int_{\alpha \in [-\pi, \pi]}^{\oplus} L_{\alpha}^2([-\pi, \pi]);$$

the Hilbert spaces L_{α}^2 appearing here are all the same, but the index reflects the fact that they arise as the closures of the space of continuous (or H^1) functions u which satisfy $u(\pi) = e^{i\alpha} u(-\pi)$. The reason for this distinction will be evident soon.

The importance of this decomposition for us is that the operator $L_{\tau,0}$ reduces further into a direct integral of operators $H_{0,\alpha}(\tau)$ induced on each one of the spaces L_{α}^2 . More specifically, $H_{0,\alpha}(\tau)$ has the same expression as $L_{\tau,0}$, but its domain is the space of H^2 functions on $[-\pi, \pi]$ satisfying the quasiperiodic boundary conditions $u(\pi) = e^{i\alpha} u(-\pi)$, $u'(\pi) = e^{i\alpha} u'(-\pi)$.

As an aside about notation, we will be considering the reduction of the full operator L_{τ} to the subspace of α -quasiperiodic functions, and shall denote this operator by $H_{\alpha}(\tau)$. Its restriction to the j^{th} eigenspace of the cross-sectional operator is denoted $H_{j,\alpha}(\tau)$, and since

$$H_{j,\alpha}(\tau) = H_{0,\alpha}(\tau) + s_{\tau}^2 j^2, \quad (19)$$

it will suffice at first to consider only the case $j = 0$. Notice that the subscripts j and α refer to the symmetries of this reduced subspace (i.e. j^{th} eigenvalue on the cross-section and quasiperiodicity α); the parameter τ has been elevated to ground level, since our ultimate concern is with the spectral flow of these operators as a function of τ .

Continuing the main thread of the discussion, even though the spaces L_{α}^2 are the same as one another, what differs is the domains of the restrictions of $L_{\tau,0}$ to each of them. Since these are self-adjoint boundary conditions, each $H_{0,\alpha}(\tau)$ has discrete spectrum:

$$\text{spec}(H_{0,\alpha}(\tau)) = \{\lambda_0(\tau, \alpha) \leq \lambda_1(\tau, \alpha) \leq \dots\}.$$

The k^{th} eigenvalue map be considered as a map:

$$\lambda_k(\tau, \cdot) : S^1 \longrightarrow \mathbb{R};$$

its image is an interval $B_k(\tau)$, which is called the k^{th} band; the union of all these intervals constitutes the spectrum of $L_{\tau,0}$:

$$\text{spec}(L_{\tau,0}) = \bigcup_{k \in \mathbb{N}} B_k(\tau).$$

A bit more is known about these bands, and the ‘band functions’ λ_k . First, since $L_{\tau,0}$ is an ordinary differential operator, the multiplicity of each of its eigenspaces is precisely two. Suppose that λ is in the spectrum, i.e. $\lambda = \lambda_k(\tau, \alpha)$ for some k and α . The corresponding eigenfunction satisfies $\varphi(t + 2\pi) = e^{i\alpha} \varphi(t)$; since the coefficients of the operator are real, $\bar{\varphi}$ is in the same eigenspace and satisfies $\bar{\varphi}(t + 2\pi) = e^{-i\alpha} \bar{\varphi}(t)$. This implies that $\lambda_k(\tau, -\alpha) = \lambda_k(\tau, \alpha)$, and hence we may as well restrict $0 \leq \alpha \leq \pi$. In fact, it is known [8] that the band functions with *even* index are nondecreasing while the band functions with *odd* index are nonincreasing in this interval, so that

$$\lambda_0(\tau, 0) \leq \lambda_0(\tau, \pi) \leq \lambda_1(\tau, \pi) \leq \lambda_1(\tau, 0) \leq \lambda_2(\tau, 0) \dots$$

This means that the bands are nonoverlapping, except perhaps at their endpoints, and we have

$$\begin{aligned} B_{2k}(\tau) &= [\lambda_{2k}(\tau, 0), \lambda_{2k}(\tau, \pi)] \\ B_{2k+1}(\tau) &= [\lambda_{2k+1}(\tau, \pi), \lambda_{2k+1}(\tau, 0)]. \end{aligned}$$

Notice that no band $B_k(\tau)$ may reduce to a point for any $\tau \in \mathcal{P}$. The reason is simply that if this were the case, then the band function $\lambda_k(\tau, \alpha)$ would be constant in α , say $\lambda_k(\tau, \alpha) \equiv \lambda_k^0(\tau)$, and hence there would be an infinite dimensional space of solutions to the equation $L_{\tau,0}u = \lambda_k^0(\tau)u$, which is at odds with the fact that $L_{\tau,0}$ is an ordinary differential operator.

For any $\lambda \in \mathbb{R}$, the space of solutions of $L_{\tau,0}u = \lambda u$ is two-dimensional. When $\lambda \in \cup_{k \in \mathbb{N}} B_k(\tau)$, then the eigenspace contains two quasi-periodic (and hence bounded) functions, whereas if λ lies outside the spectrum, then the corresponding set of solutions has a basis $\{u_{\pm}\}$, where u_+ decays exponentially as $t \rightarrow -\infty$ and grows exponentially as $t \rightarrow \infty$, while u_- grows exponentially as $t \rightarrow -\infty$ and decays exponentially as $t \rightarrow \infty$.

The main task lying ahead is to understand the dependence on τ of these bands $B_k(\tau)$. While it is certainly not possible to do this explicitly, we will use a number of special geometric solutions of $L_{\tau}u = 0$, the existence and nature of which give some nontrivial information about the bands $B_k(\tau)$, $k = 0, 1, 2$.

4.3 Geometric Jacobi fields

Solutions of $\mathcal{L}_{\tau}u = 0$ (or equivalently, $L_{\tau}u = 0$) are called Jacobi fields; they correspond to infinitesimal variations of the Delaunay surface D_{τ} which preserve the mean curvature operator to second order. These solutions may not correspond to actual CMC deformations, or may correspond to such deformations only over some portion of D_{τ} . However, there is a distinguished family of Jacobi fields characterized by the fact that they correspond to global CMC deformations; these are the Jacobi fields which are polynomially bounded on D_{τ} . We now describe these.

It is possible to use the Bloch wave theory of the previous subsection, cf. [11], [7], to obtain Fredholm properties and asymptotic expansions as $t \rightarrow \pm\infty$ for solutions of $L_{\tau}u = 0$. Crucial in this development is the fact that Jacobi fields have definite exponential rates of growth. One consequence of this theory is that there is at most a finite dimensional family of Jacobi fields u which satisfy the bounds $|u|e^{-\epsilon|t|} \leq C$ for some (and in fact any) sufficiently small ϵ . Somewhat remarkably, it is possible to identify every one of these temperate Jacobi fields because all correspond to explicit CMC deformations of D_{τ} .

There are three types of global CMC deformations: those corresponding to translations of D_{τ} , which are a three-dimensional family, those corresponding to rotations of D_{τ} , which are a two-dimensional family, and a final one-dimensional family corresponding to changing the Delaunay parameter τ . The infinitesimal variations corresponding to each of these span the six-dimensional space of temperate Jacobi fields. We now describe the distinguished three-dimensional subspace of these which are bounded.

We temporarily revert to the isothermal (s, θ) coordinate system.

- There is a Jacobi field $\Phi_{\tau}^0(s)$ which corresponds to translating D_{τ} along its axis. It is obtained by projecting the constant Killing field $(0, 0, 1)$ generating this translation over the normal vector field to D_{τ} . This yields

$$\Phi_{\tau}^0 := \partial_s \sigma_{\tau}.$$

It is clear that this Jacobi field is rotationally invariant; it is also periodic in s , hence bounded. In addition, Φ_{τ}^0 has precisely two nodal domains on the portion of D_{τ} where $s \in [0, 2\pi s_{\tau}]$.

- The two Jacobi fields corresponding to translating D_τ orthogonally to its axis are given by projecting the constant Killing fields $(1, 0, 0)$ and $(0, 1, 0)$ on the unit normal field. These functions are again periodic in s ; by choosing appropriate (complex) linear combinations of them, we may assume these Jacobi fields have the form $\Phi_\tau^{\pm 1}(s)e^{\pm i\theta}$. Closer inspection gives

$$\Phi_\tau^{\pm 1} = \cosh \sigma e^{\pm i\theta} \quad \text{when } \tau \in \mathcal{P}^+,$$

and

$$\Phi_\tau^{\pm 1} = \sinh \sigma e^{\pm i\theta} \quad \text{when } \tau \in \mathcal{P}^-.$$

Notice that $\cosh \sigma$ does not change sign while $\sinh \sigma$ has precisely two nodal domains when $s \in [0, 2\pi s_\tau]$.

Remark 3 *There are three other geometric Jacobi fields, corresponding to rotating D_τ around its axis and varying the Delaunay parameter. However, these are not periodic but grow linearly, hence do not enter our considerations below. Therefore we omit further discussion of them.*

4.4 The spectral flow

We shall henceforth always reduce to α -quasiperiodic functions, and so it is no longer necessary to think in terms of the operator L_τ on the complete Delaunay surface D_τ , nor the band structure of its spectrum; rather, we focus on the operators $H_\alpha(\tau)$, which are simpler to analyze because of their discrete spectrum. The ultimate goal now is to determine when these operators become unstable, i.e. when they have nonzero index (number of negative eigenvalues). We know this does not occur when $\tau > 0$, nor when $\tau_* < \tau < 0$. In this section we analyze the spectral flow of $H_\alpha(\tau)$ as τ decreases to $-\infty$.

The eigenvalues of this operator take the form $\lambda_{kj}(\tau) = \lambda_k(\tau, \alpha) + s_\tau^2 j^2$, $j, k = 0, 1, 2, \dots$, and so this study divides into essentially two parts. First we obtain some information about the values of $\lambda_k(\tau, \alpha)$ when $k = 0, 1, 2$; following this we turn to the problem of when $\lambda_{kj}(\tau) < 0$.

We take up the first of these tasks in the next two subsections. Because $\lambda_k(\tau, \alpha)$ always lies between $\lambda_k(\tau, 0)$ and $\lambda_k(\tau, \pi)$, we focus most of the attention on functions satisfying periodic or antiperiodic boundary conditions.

4.4.1 The spectrum of $H_{0,\alpha}(\tau)$ when $\tau > 0$

Although not needed later, we first show that there is no spectral flow for the operators $H_\alpha(\tau)$ as τ increases from 0 to 1.

Proposition 2 *Suppose that $\tau \in \mathcal{P}^+$. Then the bottom of first band $B_0(\tau)$ occurs at $\lambda = -s_\tau^2$, i.e.*

$$\lambda_0(\tau, 0) = -s_\tau^2.$$

In addition, 0 is either at the top of the second band or bottom of the third, i.e.

$$\text{either } \lambda_1(\tau, 0) = 0 \quad \text{or} \quad \lambda_2(\tau, 0) = 0.$$

Proof : We have indicated that $L_\tau(\cosh \sigma e^{i\theta}) = 0$, and it satisfies periodic boundary conditions, so that $H_{1,0}(\tau)(\cosh \sigma) = 0$. By (19), this is the same as

$$H_{0,0}(\tau)(\cosh \sigma) = -s_\tau^2 \cosh \sigma.$$

But since $\cosh \sigma$ is everywhere positive, it must correspond to the ground state eigenvalue.

Next, the function $\Phi_\tau^0 = \partial_s \sigma$ is a solution of $L_\tau w = 0$, and again satisfies periodic boundary conditions; moreover, it has precisely two nodal regions in a period domain, and so it must correspond to either the second or third eigenvalue of $H_{0,0}(\tau)$, i.e. either to $\lambda_1(\tau, 0)$ or $\lambda_2(\tau, 0)$, as stated. Unfortunately, it is not clear from the evidence at hand which of these is actually the case. \square

Corollary 1 *When $\tau > 0$, the bands $B_k(\tau)$, $k \geq 3$, are strictly contained in the positive half-line \mathbb{R}^+ , while $B_2(\tau) \subset \overline{\mathbb{R}^+}$ (and is possibly contained in the open half-line). In any event, the operator $H_{0,\alpha}(\tau)$ has no spectral flow as τ varies in \mathcal{P}^+ .*

The first statement follows at once from the previous Proposition. Obviously, since $\lambda_k(\tau, \alpha) \geq -s_\tau^2$, we have $\lambda_{kj}(\tau) \geq 0$ for all $j \geq 1$, $k \geq 0$ and all $\tau \in [0, \pi]$. This proves the last statement.

4.4.2 The spectrum of $H_{0,\alpha}(\tau)$ when $\tau < 0$

In contrast to the situation for unduloids, there is more to say about the spectrum of $H_\alpha(\tau)$ for nodoids.

Proposition 3 *Let $\tau \in \mathcal{P}^-$. Then for $\alpha \in [0, \pi]$ we have*

$$-2s_\tau^2 + \left(\frac{\alpha}{2\pi}\right)^2 - \tau^2 s_\tau^2 \leq \lambda_0(\tau, \alpha) \leq + \left(\frac{\alpha}{2\pi}\right)^2 - \tau^2 s_\tau^2;$$

in addition,

$$\lambda_1(\tau, 0) = -s_\tau^2, \quad \lambda_2(\tau, 0) = 0.$$

Proof: First consider the identities for λ_1 and λ_2 . We know that $L_\tau(\sinh \sigma e^{i\theta}) = 0$ and this function again satisfies periodic boundary conditions, so that $H_{0,0}(\tau)(\sinh \sigma) = -s_\tau^2 \sinh \sigma$. This function has two nodal regions, hence $-s_\tau^2$ must equal either the second or third eigenvalue of $H_{0,0}(\tau)$. On the other hand, we also know that $L_\tau \partial_s \sigma = 0$, and this function is again periodic, so 0 is also in the spectrum of $H_{0,0}(\tau)$. Since $\partial_s \sigma$ again has only two nodal regions, 0 must also be either the second or the third eigenvalue. Putting these statements together shows that $-s_\tau^2$ is the second eigenvalue and 0 is the third eigenvalue of $H_{0,0}(\tau)$.

To obtain the bounds on the bottom eigenvalue, note that the potential in $L_{\tau,0}$ satisfies the estimates

$$-s_\tau^2(\tau^2 + 2) \leq -s_\tau^2 \tau^2 \cosh(2\sigma) \leq -s_\tau^2 \tau^2$$

since

$$\tau^2 \cosh(2\sigma) = \tau^2 + 2\tau^2 \sinh^2 \sigma,$$

and $\tau^2 \sinh^2 \sigma \leq 1$. The estimate for $\lambda_0(\tau, \alpha)$ is then straightforward by monotonicity, but cf. below for the precise form of the eigendata of these α -quasiperiodic problems when the potential is replaced by a constant. \square

We have now shown that

$$B_0(\tau) \cup B_1(\tau) = [\lambda_0(\tau, 0), \lambda_0(\tau, \pi)] \cup [\lambda_1(\tau, \pi), -s_\tau^2] \subset (-\infty, 0),$$

and

$$B_2(\tau) = [0, \lambda_2(\tau, \pi)].$$

Recalling our earlier remark that no band reduces to a point, we have $\lambda_2(\tau, \pi) > 0$ and hence $B_k(\tau) \subset \mathbb{R}^+$ when $k \geq 3$.

Remark 4 *The main conclusion of this discussion is that only the bands $B_0(\tau)$ and $B_1(\tau)$ lie in the negative half-line; all other bands are wholly contained in the positive half-line.*

It will also be necessary later to have more refined information about the behaviour of the α -quasiperiodic eigenvalues as $\tau \rightarrow -\infty$. The key observation is that $L_{\tau,0}$ converges uniformly in this limit to $L := -\partial_t^2 - 1$, and hence the spectrum of $L_{\tau,0}$ converges to that of L . Clearly $\text{spec}(L) = [-1, \infty)$, but although L has constant coefficients we may still perform the Bloch wave analysis to decompose this spectral ray into an infinite union of spectral bands. In fact, the solutions of $Lu = \lambda u$ with $u(2\pi) = e^{i\alpha}u(0)$, $u'(2\pi) = e^{i\alpha}u'(0)$ are given by $e^{i((\alpha/2\pi) \pm k)t}$, and so for $\alpha \in [0, \pi]$, we have

$$\lambda_k(\alpha) = \begin{cases} \left(\frac{\alpha}{2\pi} + k\right)^2 - 1 & \text{when } k \text{ is even} \\ \left(\frac{\alpha}{2\pi} - k\right)^2 - 1 & \text{when } k \text{ is odd,} \end{cases}$$

and the corresponding eigenfunctions are given by

$$\varphi_k(\alpha) = \begin{cases} e^{i(\frac{\alpha}{2\pi} - k)t} & \text{when } k \text{ is even} \\ e^{i(\frac{\alpha}{2\pi} + k)t} & \text{when } k \text{ is odd.} \end{cases}$$

This gives the band structure

$$B_0 = [-1, -3/4], \quad B_1 = [-3/4, 0], \quad B_2 = [0, 5/4], \quad \text{etc.}$$

The lack of smoothness in the band functions $\lambda_k(\alpha)$ at $\alpha = 0, \pi$ is due to the absence of gaps between these bands. It is possible to find smooth band functions, but we omit this detail since it is not needed.

We now use Lemma 1 to perturb off this limiting situation.

Proposition 4 *For $\alpha \in [0, \pi]$, let $(\varphi_k(\alpha), \lambda_k(\alpha))$ and $(\varphi_k(\tau, \alpha), \lambda_k(\tau, \alpha))$ denote the eigenfunctions and eigenvalues for L and $L_{\tau,0}$, respectively. Then for τ sufficiently negative,*

$$\lambda_k(\tau, \alpha) = \lambda_k(\alpha) + \mathcal{O}(|\tau|^{-2}), \quad \partial_\tau \lambda_k(\tau, \alpha) = \mathcal{O}(|\tau|^{-3}), \quad (20)$$

and moreover,

$$\varphi_k(\tau, \alpha) = \varphi_k(\alpha) + \mathcal{O}(|\tau|^{-2}) \quad (21)$$

uniformly along with all derivatives on $0 \leq t \leq 2\pi$.

Proof: The fact that the limits of $\lambda_k(\tau, \alpha)$ and $\varphi_k(\tau, \alpha)$ converge to $\lambda_k(\alpha)$ and $\varphi_k(\alpha)$ is clear from general theory. To get the estimate on the eigenvalue, use Lemma 1 to get that

$$s_\tau^2 \tau^2 \cosh(2\sigma(s_\tau t)) = 1 + \mathcal{O}(|\tau|^{-2}), \quad \text{and} \quad \partial_\tau (s_\tau^2 \tau^2 \cosh(2\sigma(s_\tau t))) = \mathcal{O}(|\tau|^{-3}). \quad (22)$$

Assuming that the functions $\varphi_k(\tau, \alpha)$ are normalized to have L^2 norm equal to 1, we use the standard formula from eigenvalue perturbation theory

$$\partial_\tau \lambda_k(\tau, \alpha) = \langle \partial_\tau L_{\tau,0} \varphi_k(\tau, \alpha), \varphi_k(\tau, \alpha) \rangle_{L^2} = \mathcal{O}(|\tau|^{-3}),$$

which follows directly from (22). Integrating from $-\infty$ to τ gives the first part of (20); then a standard perturbation argument yields (21). \square

4.4.3 Spectral flow of $H_\alpha(\tau)$

We now let $H_\alpha(\tau)$ denote the operator L_τ acting on α -quasiperiodic functions. (Thus it is rotational invariant, and its reductions to the eigenspaces of the cross-section are the operators $H_{j,\alpha}(\tau)$.) The motivation for this whole paper is the fundamental observation that $H_\alpha(\tau)$ has a nontrivial spectral flow as τ decreases from 0 to $-\infty$. To see this, recall that we have already determined that its eigenvalues are of the form

$$\lambda_k(\tau, \alpha) + s_\tau^2 j^2, \quad j, k = 0, 1, 2, \dots$$

In addition, when $\alpha \in [0, \pi]$, $\lambda_0(\tau, \alpha)$ stays strictly negative and uniformly bounded away from zero as $\tau \rightarrow -\infty$, and the same is true for $\lambda_1(\tau, \alpha)$ for $\alpha \in [0, \pi]$ (but note that $\lambda_1(\tau, 0) = -s_\tau^2 \rightarrow 0$). In particular, we see that $\lambda_0(\tau, \alpha) + s_\tau^2 j^2$ is positive for all j (because of (11)), but eventually becomes negative for each fixed j . Thus more and more eigenvalues, corresponding to higher and higher eigenmodes on the cross-section, change signs from positive to negative.

The following two propositions make this more precise. The first is valid for any $j \geq 2$ and for ‘intermediate’ values of τ . The second gives a much more accurate estimate, but is only valid when $|\tau|$ is large.

Proposition 5 *Let $j \geq 2$. If the interval $B_0(\tau) + j^2 s_\tau^2$ contains 0, then $B_0(\tau) + 4j^2 s_\tau^2$ is entirely contained in the positive axis $(0, \infty)$.*

Proof: It follows from Proposition 3 that

$$\lambda_0(\tau, \alpha) \geq -(\tau^2 + 2) s_\tau^2.$$

Hence, $B_0(\tau) + j^2 s_\tau^2$ cannot contain 0 if $\tau^2 < 2$ and $j \geq 2$. Therefore, we can assume that $\tau < -\sqrt{2}$.

Now, suppose the result were to fail. Then for some j we would have both

$$\lambda_0(\tau, \pi) + j^2 s_\tau^2 \geq 0 \quad \text{and} \quad \lambda_0(\tau, 0) + 4j^2 s_\tau^2 < 0.$$

This gives

$$\lambda_0(\tau, 0) - 4\lambda_0(\tau, \pi) \leq 0.$$

On the other hand, from Proposition 3,

$$-(\tau^2 + 2) s_\tau^2 \leq \lambda_0(\tau, 0) < \lambda_0(\tau, \pi) \leq \frac{1}{4} - \tau^2 s_\tau^2,$$

which implies that

$$\lambda_0(\tau, 0) - 4\lambda_0(\tau, \pi) \geq -1 + 3\tau^2 s_\tau^2 - 2s_\tau^2.$$

Combining these two inequalities we get

$$s_\tau^2 \leq \frac{1}{3\tau^2 - 2}.$$

But from (10),

$$s_\tau^2 \geq \frac{1}{1 + \tau^2}.$$

These two inequalities are incompatible when $\tau < -\sqrt{2}$. □

4.5 The index of L_τ

We now apply the information we have about the spectrum of $H_\alpha(\tau)$ to the spectral flow and index of L_τ restricted to the subspace functions which are $T_{j,\alpha}$ -symmetric and also invariant under the reflection $(t, \theta) \mapsto (-t, -\theta)$. Equivalently, we let L_τ act on functions u which satisfy

$$u(t + 2\pi, \theta) = u(t, \theta + \alpha), \quad u(t, \theta + 2\pi/j) = u(t, \theta) \quad \text{and} \quad u(-t, -\theta) = u(t, \theta)$$

for all $(t, \theta) \in [-\pi, \pi] \times S^1$. As before, we denote the H^2 closure of this space by $H_{j,\alpha}^2$.

Observe that the Fourier decomposition of any function $u \in H_{j,\alpha}^2$ reads

$$u(t, \theta) = \sum_{n \in \mathbb{Z} - \{0\}} u_n(t) e^{i n j \theta}$$

where u_n satisfy $u_n(t + 2\pi) = e^{n j \alpha} u_n(t)$ and $u_n(-t) = -u_{-n}(t)$.

It is easy to see that the spectrum of L_τ acting on the above space is given by

$$\text{spec}(L_\tau) = \bigcup_{k \geq 0} \bigcup_{n \geq 1} (s_\tau^2 n^2 j^2 + B_k(\tau)).$$

Now, for $j \geq 2$, $\tau \in \mathcal{P}^-$ and $|\alpha| \leq \pi/j$, define $I_{j,\alpha}(\tau)$ to be the number of negative eigenvalues of the operator L_τ acting on $H_{j,\alpha}^2$. We have established that there exists a $\tau_* < 0$ such that $I_{j,\alpha}(\tau) = 0$ when $\tau \in (\tau_*, 0)$. As τ decreases, τ_* is the first value at which an eigenvalue of $H(\tau)$ crosses zero; since we are restricting to some proper subspace of functions, the spectrum of $H_{j,\alpha}(\tau)$ may remain positive on a larger interval. Thus define

$$\tau_{j,\alpha} = \inf \{ \tau : I_{j,\alpha}(\tau') = 0 : \tau' \in (\tau, 0) \};$$

hence all eigenvalues of $H_{j,\alpha}(\tau)$ remain positive when $\tau \in (\tau_{j,\alpha}, 0)$. So, for all $\tau \in (\tau_{j,\alpha}, 0)$, the index of L_τ is 0 and for some τ which is slightly smaller than $\tau_{j,\alpha}$, the index of L_τ is at least 1. Observe that $\tau_{j,\alpha}$ is a root of the equation $\lambda_0(\tau, j\alpha) + s_\tau^2 j^2 = 0$. The next result shows that we have a good control on $\tau_{j,\alpha}$ when j is large enough.

Proposition 6 *There exists a $j_0 \geq 2$ such that if $j \geq j_0$, and if $|\alpha| \leq \pi$, then*

$$\tau_{j,\alpha} = -\frac{j}{\sqrt{1 - (\alpha/2\pi)^2}} + \mathcal{O}(j^{-1}).$$

Moreover $I_{j,\alpha}(\tau) = 0$ for all $\tau \leq \tau_{j,\alpha}$ and $I_{j,\alpha}(\tau) \geq 1$ for all $\tau > \tau_{j,\alpha}$

Proof: The number $\tau_{j,\alpha}$ corresponds to the value of τ for which $\lambda_0(\tau, j\alpha) + s_\tau^2 j^2 = 0$. The estimate here follows from (20) and the explicit expression for $\lambda_k(\alpha)$. The point of requiring j to be large is that this forces τ also to be large, and then we can use these asymptotics results. \square

For general $j \geq 2$ our control on $\tau_{j,\alpha}$ is weaker. The proposition below is immediate from Proposition 3

Proposition 7 *For $j \geq 2$, we have $\tau_{j,0}, \tau'_{j,0} \in (-j, -\sqrt{j^2 - 2})$.*

The next result is a consequence of the proof of Lemma 1, and is the basis for proving that when j is large enough, the eigenvalue crossings are transversal.

Proposition 8 *Fix $\alpha \in [-\pi, \pi]$. There exists $j_0 \geq 2$ such that if $j \geq j_0$ and if $\lambda_k(\tau, \alpha) + s_\tau^2 j^2 = 0$, $k = 0, 1$, then*

$$\partial_\tau (\lambda_k(\tau, \alpha) + s_\tau^2 j^2) < 0.$$

Proof: First, from Lemma 1, we know that $\partial_\tau \lambda_k(\tau, \alpha) = \mathcal{O}(|\tau|^{-3})$. Also, by (13), $s_\tau^2 = 1/\tau^2 + \mathcal{O}(|\tau|^{-4})$, and $\partial_\tau s_\tau^2 = -2/\tau^3 + \mathcal{O}(|\tau|^{-5})$. Hence

$$\partial_\tau(\lambda_k(\tau, \alpha) + s_\tau^2 j^2) = -2j^2 \tau^{-3} + \mathcal{O}(|\tau|^{-3}).$$

Then observe that, since $\lambda_k(\tau, \alpha) + s_\tau^2 j^2 = 0$ (recall that λ_k is bounded since we only need consider $k = 0, 1$), then $s_\tau \sim j^{-1} \sim |\tau|^{-1}$, so $\partial_\tau(\lambda_k(\tau, \alpha) + s_\tau^2 j^2)$ is certainly negative when j is large enough. \square

5 CMC deformations of nodoids

We now employ the preceding results about the Jacobi operator to deduce the existence of new CMC surfaces with $T_{j,\alpha}$ symmetry.

5.1 The mean curvature operator

For any $\tau \in \mathcal{P}$, parametrize D_τ by

$$X_\tau(t, \theta) := \frac{1}{2} \left(\tau e^{\sigma_\tau(s_\tau t)} \cos \theta, \tau e^{\sigma_\tau(s_\tau t)} \sin \theta, \kappa_\tau(s_\tau t) \right).$$

The unit normal at $X_\tau(t, \theta)$ is then

$$N_\tau(t, \theta) := (\tau \sinh \sigma_\tau(s_\tau t) \cos \theta, \tau \sinh \sigma_\tau(s_\tau t) \sin \theta, -\partial_s \sigma_\tau(s_\tau t)). \quad (23)$$

If w is any function which is \mathcal{C}^2 small, then let D_w which by definition is the image of the map

$$X_w = X_\tau + w N_\tau.$$

Note that if w is R_j - or S_α - or $T_{j,\alpha}$ -symmetric, then D_w has the same symmetries.

There is a rather complicated nonlinear elliptic equation which determines when D_w has mean curvature 1. We do not write this out in full, but write it in abbreviated form as

$$L_\tau w + Q_\tau(w) = 0, \quad (24)$$

where L_τ is the multiple of the Jacobi operator which we have been studying, and Q_τ is a second order nonlinear differential operator which vanishes quadratically at $w = 0$. More precise information about the structure of Q_τ is given in [9]. Note that in the (t, θ) coordinate system, Q_τ is 2π periodic.

Because this is a nonlinear problem, we shall use the function spaces $\mathcal{C}_{j,\alpha}^{k,\beta}(\mathbb{R} \times S^1)$, defined for any $j \geq 2$ and $\alpha \in [-\pi/j, \pi/j]$ to contain all $\mathcal{C}^{k,\beta}$ functions which are $T_{j,\alpha}$ -symmetric and invariant under the reflection $(t, \theta) \mapsto (-t, -\theta)$:

$$\mathcal{C}_{j,\alpha}^{k,\beta}(\mathbb{R} \times S^1) = \{u \in \mathcal{C}^{k,\beta}(\mathbb{R} \times S^1) : u(t + 2\pi, \theta) = u(t, \theta + \alpha),$$

$$u(t, \theta + 2\pi/j) = u(t, \theta) \quad \text{and} \quad u(-t, -\theta) = u(t, \theta)\}.$$

Clearly

$$L_\tau : \mathcal{C}_{j,\alpha}^{2,\beta}(\mathbb{R} \times S^1) \longrightarrow \mathcal{C}_{j,\alpha}^{0,\beta}(\mathbb{R} \times S^1)$$

and

$$Q_\tau : \mathcal{C}_{j,\alpha}^{2,\beta}(\mathbb{R} \times S^1) \longrightarrow \mathcal{C}_{j,\alpha}^{0,\beta}(\mathbb{R} \times S^1)$$

are smooth.

Instead of constructing the surfaces bifurcating from D_τ as submanifolds of \mathbb{R}^3 , it is equivalent to take the quotient of \mathbb{R}^3 with respect to the screw motion S_α along the z -axis, with translation length 2π . The quotient of \mathbb{R}^3 by this motion is a 2-plane bundle over S^1 with holonomy α , and D_τ/S_α is a compact submanifold.

5.2 Bifurcations

We are now in a position to prove the existence of families of (immersed) CMC surfaces which bifurcate off the Delaunay surfaces D_τ .

The proof of Theorem 1 follows from the general bifurcation theorem of Smoller and Wasserman [15], cf. also Theorem 13.10 in [14]. To apply this result, we require only the fact that, by definition, the index $I_{j,\alpha}(\tau)$ is 0 for any $\tau < \tau_{j,\alpha}$ and, by Proposition 5, the index $I_{j,\alpha}(\tau)$ is exactly 1 for some $\tau > \tau_{j,\alpha}$, arbitrarily close to $\tau_{j,\alpha}$. This change of multiplicity ensures the existence of a bifurcation for the nonlinear problem (24) in the space $C_{j,\alpha}^{2,\beta}(\mathbb{R} \times S^1)$ (modulo the screw motion S_α). When $\alpha = 0$, this produces nonrotationally invariant CMC surfaces which are periodic and R_j -symmetric. When $\alpha \neq 0$ this produces nonrotationally invariant CMC surfaces which are $T_{j,\alpha}$ -symmetric. By Proposition 7 we get the estimate on the location of this bifurcation point.

One defect of this general theorem is that we obtain no information about whether this bifurcation gives a smooth connected branch of solutions. For this we require nondegenerate crossing of a simple eigenvalue. However, in Proposition 8, we have verified this hypothesis when j and hence τ is sufficiently large. Therefore, we obtain Theorem 2 from the theorem of Crandall and Rabinowitz [1], cf. also Theorem 13.5 in [14].

We conclude by noting that there are many unresolved questions concerning the surfaces we have produced. The most obvious one concerns the existence of a second bifurcation which arises when $\lambda_1(\tau, \alpha) + s_\tau^2 j^2$ crosses zero. This should not be difficult to obtain, and requires only a slight elaboration of the techniques and estimates we have been using. One complication here is how to separate the second bifurcation for some smaller value of j occurring at the same value of τ as the first bifurcation for a larger value of j . We have not pursued this because our information about these surfaces is so limited, since we have proved their existence using an abstract functional analytic technique. The most interesting problem is to globalize this construction and find a complete characterization of all cylindrically bounded CMC surfaces with two ends.

References

- [1] M. Crandall and P. Rabinowitz, *Bifurcation, perturbation of simple eigenvalues and linearized stability*, Arch. Rational Mech. Anal. 52 (1973), 161–180.
- [2] C. Delaunay, *Sur la surface de revolution dont la courbure moyenne est constante*, Jour. de Mathématiques, 6 (1841), 309-320.
- [3] J. Eells, *The surfaces of Delaunay*. Math. Intelligencer 9 (1987), no. 1, 53–57.
- [4] N. Korevaar and R. Kusner, *The global structure of constant mean curvature surfaces*, Invent. Math. 114 (1993), 311-332.
- [5] N. Korevaar, R. Kusner, and B. Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom. 30 (1989), no. 2, 465–503.
- [6] K. Grosse-Brauckmann, R. Kusner and J. Sullivan, *Constant mean curvature surfaces with three ends*, MSRI preprint # DG/9903101.

- [7] R. Kusner, R. Mazzeo and D. Pollack, *The moduli space of complete embedded constant mean curvature surfaces*, *Geom. Funct. Anal.* **6** (1996) 120–137.
- [8] W. Magnus and S. Winkler *Hill's equation*, (1979), Dover Press, Toronto.
- [9] R. Mazzeo and F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, *Comm. Anal. Geom.* **9** No. 1 (2001) 169–237.
- [10] R. Mazzeo, F. Pacard and D. Pollack, *The conformal theory of Alexandrov embedded constant mean curvature surfaces in \mathbb{R}^3* , preprint.
- [11] R. Mazzeo, D. Pollack and K. Uhlenbeck, *Moduli spaces of singular Yamabe metrics*, *J. Amer. Math. Soc.* **9** (1996), no. 2, 303–344.
- [12] W. Meeks III, *The topology and geometry of embedded surfaces of constant mean curvature*, *J. Differential Geom.* **27** (1988), no. 3, 539–552.
- [13] M. Reed and B. Simon, *Methods of modern mathematical physics, Vol. IV*, Academic Press (1979).
- [14] J. Smoller, *Shock waves and reaction-diffusion equations*, Second Ed. (1994), Grundlehren der mathematischen Wissenschaften **258**, Springer-Verlag, New York.
- [15] J. Smoller and A. Wasserman, *Bifurcation and symmetry-breaking*, *Invent. Math.* **100** (1990), no. 1, 63–95.