ANALYTIC CONTINUATION OF THE RESOLVENT OF THE LAPLACIAN ON SYMMETRIC SPACES OF NONCOMPACT TYPE

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ABSTRACT. Let (M, g) be a globally symmetric space of noncompact type, of arbitrary rank, and Δ its Laplacian. We prove the existence of a meromorphic continuation of the resolvent $(\Delta - \lambda)^{-1}$ across the continuous spectrum to a Riemann surface multiply covering the plane. The methods are purely analytic and are adapted from quantum N-body scattering.

1. INTRODUCTION

A basic problem in geometric scattering theory is to carry out a refined analysis of the resolvent of the Laplacian on various classes of complete manifolds with regular geometry at infinity. The symmetric spaces of noncompact type comprise a natural class of manifolds to understand from this point of view because their asymptotic geometry is so well understood. An added attraction is that the analytic properties of the Laplacians on these spaces are closely connected to representation theory and number theory. In this paper we continue our program, initiated in [14], to extend the methods and results of geometric scattering theory to this setting. More specifically, let $M = G/K$ be a symmetric space of noncompact type, with $\mathrm{rank}(M) = n$, and denote by $\Delta = \Delta_M$ its Laplace-Beltrami operator with respect to some choice of invariant metric. We do not assume that M is irreducible, so any such metric is obtained by fixing a constant multiple of the Killing form on each irreducible factor. As M is complete, Δ is self-adjoint. The resolvent of the Laplacian is the operator $R(\lambda) = (\Delta - \lambda)^{-1}$, initially defined when $\lambda \in \mathbb{C} \setminus [0, \infty)$ as a bounded operator on $L^2(M)$. In this paper we prove that $R(\lambda)$ continues meromorphically to a larger set. The existence of this continuation is classical when M is a Euclidean space, and is also well known for rank one symmetric spaces and their geometric generalizations, e.g. conformally compact spaces [12] and their complex analogues [3]; it is also known in the case of higher rank *complex* symmetric spaces, but surprisingly, its existence for higher rank *real* symmetric spaces is only known indirectly [4]. Recently we used techniques from microlocal analysis to prove this continuation in the two simplest rank 2 situations: when M is a product of hyperbolic spaces [14] and when $M = \mathrm{SL}(3)/\mathrm{SO}(3)$ [15], [13], and our goal in this paper is to extend that construction to the general case. Let $G_o(\lambda)$ denote the Green function, i.e. the Schwartz kernel of $R(\lambda)$. This is our main result:

Theorem 1.1. *The Green function $G_o(\lambda)$ continues meromorphically as a distribution to a Riemann surface $\widetilde{\mathcal{Y}}_{\pi/2}$ (see Definition 5.8), ramified at a sequence of*

Date: August 5, 2003.

points corresponding to translates of the poles of the meromorphic continuation of $G_o(\lambda)$ on symmetric spaces of lower rank.

It is then natural to ask whether these poles exist. Here we show that they lie in a compact set in the complement of any cone containing a singular direction; in fact, an estimate which implies this plays an important role in the proof of the existence of the continuation. However, we conjecture that this continuation has no poles at all on $\widetilde{\mathcal{Y}}_{\pi/2}$; see the remark at the end of the last section.

We sketch part of $\widetilde{\mathcal{Y}}_{\pi/2}$ in Figure 1. The thick line in picture on the left shows the spectrum of Δ (inside $\mathbb{C}$). It is a half line $[\lambda_0, \infty)$, and the Green function $G_o(\lambda)$ is defined a priori for $\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)$. The thin line indicates the rest of the real axis.

The picture on the right shows the analytic continuation of $G_o(\lambda)$ across the ray $[\lambda_0, \infty)$ from below; it is defined *outside* the thick half lines, one of which is $(-\infty, \lambda_0]$. The thin line is again the rest of the real axis. Thus, for $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda < 0$, $G_o(\lambda)$ is defined identically in the two pictures, but for $\operatorname{Im} \lambda > 0$, on the right hand side, $G_o(\lambda)$ lives on a different sheet of the Riemann surface, whose projection to $\mathbb{C}$ is shown. The ramification points are indicated by the thickened points; the conjecture then is that none of these exist except λ_0.

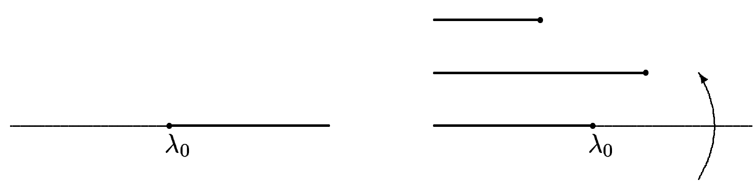

FIGURE 1. Part of the domain of the analytic continuation of $G_o(\lambda)$.

We proceed by induction on the rank of the symmetric space. The two key ingredients of the proof are complex scaling, and the construction of a parametrix, i.e. an approximate inverse, for the complex scaled K-radial Laplacian. This method is closely related to the analogous problem in N-body scattering, where it was introduced by Balslev and Combes [1] and extended by C. Gérard [5]. Indeed, technically the only reason why we cannot use the N-body results directly is that if we identify Δ acting on K-invariant functions with a differential operator on a flat $\mathcal{A} = \exp(\mathfrak{a})$, and hence on $\mathfrak{a}$, the L^2 space on $\mathfrak{a}$ is not the Euclidean one, and the first order terms are singular at the walls of the Weyl chambers.

Complex scaling in this setting is induced by dilation along geodesic rays from o. These are the maps Φ_θ that, for $\theta \in \mathbb{R}$, send any point $\gamma(t)$ on any geodesic γ with $\gamma(0) = o$ to the point $\gamma(e^\theta t)$. Complex scaling extends these analytically in θ to a domain in the complex plane. The virtue of the scaling is that, for complex values of θ, the essential spectrum of the scaled radial Laplacian is (almost) a rotation of the essential spectrum of the Laplacian, and this allows the analytic continuation of the resolvent. We define and describe the scaling here in §5, and we refer to the introduction of [13] for a brief description of this procedure for the Laplacian on the hyperbolic plane.

Although the other ingredient, the parametrix construction, is fundamentally microlocal, we minimize the explicit use of microlocal techniques, which is possible because of the essentially 'soft' nature of such an analytic continuation result, and because there are finitely many local 'product models' for the scaled radial Laplacian $\Delta_{\mathrm{rad},\theta}$, i.e. locally (in certain neighbourhoods of infinity) this operator has the form $A \otimes \mathrm{Id} + \mathrm{Id} \otimes B$ modulo decaying error terms. More delicate questions concerning the precise asymptotic behaviour of the Green function may be approached using an elaboration of the same construction, as in [14], [15], but do require more attention to the microlocal aspects; we shall return to this elsewhere.

Although our analysis seems to make essential use of various compactifications of M, in fact these are not truly essential. Rather, they are very helpful in the construction of certain partitions of unity, on the support of which $\Delta_{\mathrm{rad},\theta}$ is particularly well approximated by product models. Such partitions of unity could also be described by requiring various homogeneity properties, but in the further development of the scattering theory on symmetric spaces, e.g. in the study of the asymptotics of the Green function, these compactifications play a central role.

We would also like to underline that it is crucial that the product models for $\Delta_{\mathrm{rad},\theta}$ are valid in *conic* subsets of $\mathfrak{a}$ – in the language of compactifications, this is the reason why we use a partition of unity and cutoffs on the radial (or geodesic) compactification $\hat{\mathfrak{a}}$. The conic cutoffs give decaying error terms in the parametrix construction; this would not be the case if we localized at finite distances from Weyl chamber walls.

Finally, this work would not be complete without commenting on its relationship to the meromorphic continuation of Harish-Chandra's c-function. The c-function is known to have a meromorphic extension to the flat $\mathfrak{a}^*_{\mathbb{C}}$, and its restriction to the vectors in $\mathfrak{a}^*$ with length $\sqrt{\lambda - \lambda_0}$ can be thought of as a 'scattering matrix' by analogy with both the rank-one case and N-body scattering. Now, in the latter settings, the poles of the meromorphic continuation of the scattering matrix (considered as an operator) and the resolvent coincide – one might expect that if the c-function is analytic on the rotation of this sphere around 0 in $\mathfrak{a}^*_{\mathbb{C}}$ by angle $\arg\sqrt{\lambda - \lambda_0}$, then the continuation of the resolvent does not have a pole at λ, and conversely. There is an explicit formula for the c-function, see [8, Chapter IV, Theorem 6.14], and it is apparent from it that this requirement on the c-function is never satisfied in the higher rank setting (since some inner product may vanish). This phenomenon already occurs in the complex case, when the formula is simpler [8, Chapter IV, Theorem 5.7]. We expect, however, that, suitably renormalized and considered as an operator, the meromorphic structure of the c-function can be related to that of the resolvent.

We are very grateful to Gilles Carron, Sigurdur Helgason, Lizhen Ji, Richard Melrose, David Vogan and Maciej Zworski for helpful discussions and their encouragement. R. M. is partially supported by NSF grant #DMS-0204730; A. V. is partially supported by NSF grant #DMS-0201092 and a Fellowship from the Alfred P. Sloan Foundation.

2. Compactifications of $\mathfrak{a}$ and the radial Laplacian

In this section, we begin by reviewing some well-known facts about the Lie-theoretic algebra and global geometry of the symmetric space M; we refer to [7], [8] for a comprehensive development and all proofs, and also to [2] for a detailed summary from a more geometric point of view. Of central importance here is the flat $\mathcal{A} = \exp(\mathfrak{a})$; $\mathfrak{a}$ is a Euclidean space of dimension $\text{rank}(M)$, and it is the ultimate locus of our analysis. We shall systematically identify $\mathfrak{a}$ with its exponential, and will usually work on $\mathfrak{a}$ rather than $\mathcal{A}$, since it is more customary to use linear coordinates rather than their exponentials. We go on to define two compactifications of this flat, $\overline{\mathfrak{a}}$ and the larger one $\widetilde{\mathfrak{a}}$, which play a central role in our approach. Motivation for these definitions is provided by the specific form of the radial Laplacian Δ_{rad} on M, which is introduced and discussed along the way. We conclude by showing that the radial Laplacian on symmetric spaces of lower rank appear in the restrictions of this operator to boundary faces of $\widetilde{\mathfrak{a}}$.

2.1. **Geometry of flats.** Suppose $M = G/K$, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Thus $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ its orthogonal complement with respect to the Kiling form, which is identified with T_oM (o will always denote the identity coset). We also fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$; this is always of the form $\mathfrak{p} \cap \mathfrak{g}_0$, where $\mathfrak{g}_0$ is a maximal abelian subalgebra (called a Cartan subalgebra) in $\mathfrak{g}$, and conversely, any such intersection is a maximal abelian subspace in $\mathfrak{p}$. The number $n := \dim \mathfrak{a}$ is called the rank of M, and $\exp \mathfrak{a} := \mathcal{A}$ is a totally geodesic flat submanifold which is maximal with respect to this property, and is called a flat. It is isometric to $\mathbb{R}^n$.

A key example, to which we shall refer back repeatedly throughout this paper for purposes of illustration, is $\mathcal{M}_{n+1} = \text{SL}(n+1)/\text{SO}(n+1)$. Here $\mathfrak{g} = \mathfrak{sl}(n+1)$ consists of all $(n+1)$-by-$(n+1)$ matrices of trace zero, and $\mathfrak{k} = \mathfrak{so}(n+1)$ and $\mathfrak{p}$ consist of all such matrices which are skew-symmetric, respectively symmetric. We may take $\mathfrak{a}$ to be the subspace of diagonal matrices of trace zero. Denoting these diagonal entries by t_i, $i = 1, \ldots, n+1$, then the diagonal matrices A_i, $i = 1, \ldots, n$, with $t_i = 1$, $t_{i+1} = -1$ and all other $t_j = 0$ comprise the standard basis of $\mathfrak{a}$. We identify $\mathcal{M}_{n+1}$ with the space of positive definite symmetric matrices via the identification $\text{SL}(n+1) \ni B \to \sqrt{B^tB}$. The flat $\mathcal{A} = \exp(\mathfrak{a})$ consists of diagonal matrices with positive entries $\lambda_1, \ldots, \lambda_{n+1}$ and determinant 1.

Since $\mathfrak{a}$ is abelian, there is a simultaneous diagonalization for the commuting family of symmetric homomorphisms ad H, $H \in \mathfrak{a}$, on $\mathfrak{g}$. A simultaneous eigenvector X satisfies $(\text{ad } H)(X) = \alpha(X)$ for every $H \in \mathfrak{a}$, for some element $\alpha \in \mathfrak{a}^*$; the set of linear forms which arise in this way constitute the (finite) set of (restricted) roots Λ for $\mathfrak{g}$, and the space of eigenvectors associated to each $\alpha \in \Lambda$ is the 'root space' $\mathfrak{g}_\alpha$. Thus in particular $0 \in \Lambda$ and its root space $\mathfrak{g}_0$ is the Cartan subalgebra above (i.e. if we fix $\mathfrak{a}$ first, then a Cartan subalgebra is uniquely associated in this way), and $\mathfrak{g} = \oplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha$. We shall always use the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{p}$ as the inner product $\langle \cdot, \cdot \rangle$ (rather than allowing for different scalar multiples of the Killing form on different factors in a decomposition into irreducible subalgebras). This determines the root vectors $H_\alpha \in \mathfrak{a}$ by the relationship $\alpha(H) = \langle H, H_\alpha \rangle$ for all $H \in \mathfrak{a}$. We also fix a partition $\Lambda = \Lambda^+ \cup \Lambda^-$, $\Lambda_- = -\Lambda_+$, into positive and negative roots. There is a subset $\Lambda^+_{\text{ind}} \subset \Lambda^+$ of indecomposible (or simple) positive roots which is a basis for $\mathfrak{a}^*$ (so in particular, $\#\Lambda^+_{\text{ind}} = n$) such that for any $\alpha \in \Lambda$,

$$\alpha = \sum_{\alpha_j \in \Lambda^+_{\text{ind}}} n_j \alpha_j, \qquad \text{where all } n_j \in \mathbb{Z} \quad \text{and} \quad \begin{cases} \text{all } n_j \geq 0 & \text{if } \alpha \in \Lambda^+ \\ \text{all } n_j \leq 0 & \text{if } \alpha \in \Lambda^-. \end{cases}$$

Of particular importance is the element

$$\rho = \frac{1}{2} \sum_{\alpha \in \Lambda^+} m_\alpha \, \alpha \in \mathfrak{a}^*, \tag{2.1}$$

where $m_\alpha = \dim \mathfrak{g}_\alpha$, and its metrically dual vector $H_\rho \in \mathfrak{a}$.

Each $\alpha \in \Lambda$ determines a hyperplane $W_\alpha = \alpha^{-1}(0) \subset \mathfrak{a}$, called the Weyl chamber wall associated to α, and by definition

$$\mathfrak{a}_{\mathrm{reg}} = \mathfrak{a} \setminus \left(\bigcup_{\alpha \in \Lambda} W_\alpha \right)$$

is called the set of *regular* vectors; the components of this set are called (open) Weyl chambers, and the distinguished component

$$C^+ = \{H \in \mathfrak{a} : \alpha(H) > 0, \quad \forall \, \alpha \in \Lambda^+\},$$

is called the positive Weyl chamber. We also define

$$W_{\alpha,\mathrm{reg}} = W_\alpha \setminus \left(\bigcup_{\beta \neq \alpha} (W_\beta \cap W_\alpha) \right).$$

As already indicated, we shall systematically identify each of these sets with their corresponding exponentials in $\mathcal{A}$: in particular, set $\mathcal{A}_{\mathrm{reg}} = \exp(\mathfrak{a}_{\mathrm{reg}})$, $\exp(W_\alpha) = \mathcal{W}_\alpha$, $\mathcal{W}_{\alpha,\mathrm{reg}} = \exp(W_{\alpha,\mathrm{reg}})$ and $\exp(C^+) = \mathcal{C}^+$.

The orthogonal reflections across the Weyl chamber walls generate a finite group, called the Weyl group W. Alternately, W is the quotient $N(\mathfrak{a})/Z(\mathfrak{a})$ of the normalizer by the centralizer of $\mathfrak{a}$ with respect to the adjoint action Ad of K on $\mathfrak{g}$. The Weyl group acts simply transitively on the set of Weyl chambers.

Returning again to the special case $M = \mathcal{M}_{n+1}$, the root set Λ consists of all α_{ij}, where for the diagonal matrix $T = \mathrm{diag}(t_1, \dots, t_{n+1})$, $\alpha_{ij}(T) = t_i - t_j$. We take $\Lambda^+ = \Lambda^+_{\mathrm{ind}} = \{\alpha_{i+1\, i}, 1 \le i \le n\}$; so that the positive Weyl chamber C^+ consists of all traceless diagonal matrices A with all $t_1 < t_2 \ldots < t_{n+1}$, while $\mathcal{C}^+$ consists of all unimodular diagonal matrices such that $0 < \lambda_1 < \ldots < \lambda_{n+1}$. The centralizer $Z(\mathfrak{a})$ in $\mathrm{SO}(n+1)$ is the set of diagonal matrices with entries equal to ± 1, while the normalizer $N(\mathfrak{a})$ in $\mathrm{SO}(n+1)$ is the set of signed permutation matrices, and so the Weyl group W is identified with the symmetric group S_{n+1}, and acts by permutations on the entries of the diagonal matrices.

G acts on $M = G/K$ by left multiplication. The Cartan decomposition states that $G = K \cdot \mathcal{A} \cdot K$, and in stronger form, $G = K \cdot \overline{\mathcal{C}^+} \cdot K$. Moreover, for $g \in G$, with $g = k_1 a k_2$, the element $a \in \overline{\mathcal{C}^+}$, as well as $H \in C^+$ satisfying $a = \exp H$, are uniquely determined; we write $H = H(g)$. This induces a map on M, so for $p = gK \in M$, $H(p) = H(g)$.

The geodesic exponential map $\exp : \mathfrak{p} \to M$ is a diffeomorphism. Moreover, $k \cdot \exp(X) = \exp(\mathrm{Ad}(k)X)$ for $k \in K$, $X \in \mathfrak{p}$.

Letting $G_{\mathrm{reg}} = K\mathcal{A}_{\mathrm{reg}}K = K\mathcal{C}^+K$ and $M_{\mathrm{reg}} = G_{\mathrm{reg}} \cdot o$, then M_{reg} is diffeomorphic to $K' \times \mathcal{C}^+$, where $K' = K/Z(\mathcal{A})$, see [7, Ch. IX, Corollary 1.2]. In fact, K' acts freely on $\mathcal{A}_{\mathrm{reg}}$, but if $X \in \mathcal{A} \setminus \mathcal{A}_{\mathrm{reg}}$, then the isotropy group $K^X \subset K$ is strictly larger than $Z(\mathcal{A})$. Fixing a root α, then all the isotropy groups K^X for $X \in \mathcal{W}_{\alpha,\mathrm{reg}}$ are the same, and we denote this common group by K^α. There is a larger subgroup $K^{\mathcal{W}} \subset K$ which maps $\mathcal{A} \setminus \mathcal{A}_{\mathrm{reg}}$ to itself (and hence permutes the Weyl chamber walls). The entire symmetric space is obtained as the quotient of $K' \times \overline{\mathcal{C}^+}$ by the diagonal Weyl group action.

Following the last paragraph, we see that elements of $\mathcal{C}^\infty(M)^K$, the space of smooth K-invariant functions on M, restrict to elements of $\mathcal{C}^\infty(\mathcal{A})^W$, the space of smooth W-invariant functions on $\mathcal{A}$; we later show in Proposition 3.1 that this map is an isomorphism. More generally, we shall use the notation that if E is any space of functions (on M or $\mathcal{A}$ or any other related space) and if Γ is a group on the underlying space, then E^Γ is the subspace of Γ-invariant elements.

2.2. **The radial Laplacian.** Before proceeding with further geometric considerations, we now introduce the radial Laplacian Δ_{rad}, which is simply the restriction the full Laplacian Δ_M to K-invariant functions (or distributions) on M. Δ_{rad} is our principal object of study in this paper, and the main task ahead of us is the construction of parametrices for $(\Delta_{\mathrm{rad}} - \lambda)^{-1}$.

Rather than thinking of the radial Laplacian as an operator on M, acting on a restricted space of functions, it is more useful to realize Δ_{rad} as an operator acting on essentially arbitrary functions on a lower dimensional manifold. This is done by restricting to functions on a submanifold transverse to the orbits of K on M, and the simplest choice is to restrict to the regular part of the flat $\mathcal{A}_{\mathrm{reg}}$, which we identify with $\mathfrak{a}_{\mathrm{reg}}$. Of course, we will then have to investigate the extension of this operator to the entire flat.

There is an elegant expression for the radial Laplacian on $\mathfrak{a}_{\mathrm{reg}}$:

$$\Delta_{\mathrm{rad}} = \Delta_{\mathfrak{a}} + \frac{1}{2}\sum_{\alpha\in\Lambda}(m_\alpha \coth\alpha)\, H_\alpha, \tag{2.2}$$

where $\Delta_{\mathfrak{a}}$ is the standard Laplacian on the vector space $\mathfrak{a}$, $m_\alpha = \dim \mathfrak{g}_\alpha$ and H_α is the root vector associated to the root α, as defined in §2.1. Noting that $m_\alpha = m_{-\alpha}$, $\coth(-\alpha) = -\coth\alpha$ and $H_{-\alpha} = -H_\alpha$, we also have

$$\Delta_{\mathrm{rad}} = \Delta_{\mathfrak{a}} + \sum_{\alpha\in\Lambda^+}(m_\alpha \coth\alpha)\, H_\alpha, \tag{2.3}$$

which is the expression found in [8, Ch. II, Proposition 3.9]. It is clear from (2.2) that the action of W on $\mathfrak{a}_{\mathrm{reg}}$ leaves Δ_{rad} invariant. The singularities in the coefficients of these first order terms along the Weyl chamber walls might seem to complicate the process of extending this operator to all of $\mathfrak{a}$, and indeed this would be the case if we were to try to let Δ_{rad} act on $\mathcal{C}^\infty(\mathfrak{a})$, for example. However, this difficulty disappears if we restrict to W-invariant functions. Indeed, we shall prove in the next section that $\mathcal{C}^\infty(M)^K$ is naturally identified with $\mathcal{C}^\infty(\mathfrak{a})^W$, and so (tautologically) Δ_{rad} extends to this latter space, and then also to W-invariant distributions, etc. As a first step toward this identification, we prove the

Lemma 2.1. *The operator* $\Delta_{\mathrm{rad}} : \mathcal{C}^\infty(\mathfrak{a}_{\mathrm{reg}})^W \to \mathcal{C}^\infty(\mathfrak{a}_{\mathrm{reg}})^W$ *induces a map* $L : \mathcal{C}^\infty(\mathfrak{a})^W \to \mathcal{C}^\infty(\mathfrak{a})^W$ *via the inclusion* $\iota : \mathfrak{a}_{\mathrm{reg}} \hookrightarrow \mathfrak{a}$. *That is, if* $f \in \mathcal{C}^\infty(\mathfrak{a})^W$, *then* $\Delta_{\mathrm{rad}}\iota^* f = \iota^* g$ *for some* $g \in \mathcal{C}^\infty(\mathfrak{a})^W$, *and* $g = Lf$ *is uniquely determined by* f.

Proof. By the density of $\mathfrak{a}_{\mathrm{reg}}$ in $\mathfrak{a}$ and the smoothness of g, it is clear that g will be unique once we know it exists. To prove its existence, note first that $\Delta_{\mathfrak{a}}$ commutes with any reflection on $\mathfrak{a}$, hence is invariant by the action of W, and so maps $\mathcal{C}^\infty(\mathfrak{a})^W$ to itself. Thus it suffices to prove that the same is true for each of the summands

$\coth\alpha\, H_\alpha$, $\alpha \in \Lambda^+$. For any $\beta \in \Lambda^+$, let R^β denote the reflection across the wall W_β, and $\mathcal{C}^\infty(\mathfrak{a})^{R_\beta}$ the space of functions invariant by this reflection. Writing

$$\coth\alpha\, H_\alpha = (\alpha \coth\alpha)\frac{1}{\alpha} H_\alpha,$$

then, since both α and $\coth\alpha$ are simultaneously either fixed or taken to their negatives by any R_β, we have $\alpha\coth\alpha \in \mathcal{C}^\infty(\mathfrak{a})^{R_\beta}$ for every β. Thus we reduce at last to proving that for each α and β, $\alpha^{-1}H_\alpha$ maps $\mathcal{C}^\infty(\mathfrak{a})^{R_\beta}$ to itself. But $S^\alpha = W_\alpha^\perp = \operatorname{span}(H_\alpha)$ is a copy of $\mathbb{R}$ and the smooth even functions on this line are all smooth functions of $\sigma = \alpha^2$, and so the operator $\alpha^{-1}H_\alpha = 2\frac{d}{d\sigma}$ certainly preserves the space of smooth even functions. Similarly, any element $f \in \mathcal{C}^\infty(\mathfrak{a})^{R_\beta}$ can be regarded as a family of smooth even functions $\tilde{f}_x$ on S^α too, as x ranges over W_α, and the action of $\alpha^{-1}H_\alpha$ on f may be determined from the induced action on $\tilde{f}_x$.

We have proved that if $f \in \mathcal{C}^\infty(\mathfrak{a})^W$, then there is a function $Lf \in \mathcal{C}^\infty(\mathfrak{a})$ which agrees with $\Delta_{\mathrm{rad}} f$ on $\mathfrak{a}_{\mathrm{reg}}$; the W-invariance of Lf follows from its W-invariance on the dense subset $\mathfrak{a}_{\mathrm{reg}}$. □

The actual identification of $\mathcal{C}^\infty(M)^K$ with $\mathcal{C}^\infty(\mathfrak{a})^W$ uses this lemma, but also requires the ellipticity of Δ_M, and so we defer the proof until we have covered more preliminaries. However, we emphasize the conclusion, that the singularities of Δ_{rad} are of the same nature as the singularities of the Laplacian on $\mathbb{R}^n$ when written in in polar coordinates.

We conclude this subsection by exhibiting the many-body structure of Δ_{rad} more plainly. Write

$$\Delta_{\mathrm{rad}} = \Delta_{\mathfrak{a}} + 2H_\rho + E, \tag{2.4}$$

where H_ρ is as in (2.1), and

$$E = \sum_{\alpha\in\Lambda^+} m_\alpha(\coth\alpha - 1)H_\alpha.$$

The first terms, $\Delta_{\mathfrak{a}} + 2H_\rho$, are translation invariant, hence can be analyzed easily using Fourier analysis. On the other hand, each summand in E is a first order operator which decays exponentially as the corresponding root $\alpha \to +\infty$. This rearrangement of the first order terms is only satisfactory in C^+, but the W invariance of Δ_{rad} implies that it is meaningful everywhere. The vectors H_α are not independent (except in the special, completely reducible case), and so (2.4) shows that Δ_{rad} has first order interaction terms of N-body type, where the finite intersections of Weyl chamber walls play the role of 'collision planes'.

2.3. **Compactifications.** Because of the many-body structure of Δ_{rad}, any thorough analysis of this operator and its resolvent must include some sort of delicate localization at infinity. As already explained in the introduction, the traditional approach of Harish-Chandra is most effective in sectors disjoint from the Weyl chamber walls, while uniformity of behaviour of various analytic objects on approach to these walls is more difficult to obtain; on the other hand, in our approach these walls are essentially 'interior points', and create no difficulties. The main issue is to find and work in neighbourhoods which most effectively intermediate between these two types of behaviour. The use of compactifications to localize at infinity, or at least to better visualize and control these localizations, is well known. In the next subsections we shall introduce three main compactifications: the first, $\hat{\mathfrak{a}}$, is the geodesic, or radial, compactification; the second, $\overline{\mathfrak{a}}$, is known as the dual-cell compactification; the third, $\widetilde{\mathfrak{a}}$, is the minimal compactification which dominates the other two. All of these have been used elsewhere, cf. [6], [18], but we shall emphasize their smooth structures; in particular our contention (born out by the conclusions of this paper) that $\widetilde{\mathfrak{a}}$ is the most appropriate place to study Δ_{rad}, is a novel perspective.

As orientation for the remainder of §2, we sketch what lies ahead. The radial compactification $\hat{\mathfrak{a}}$ is by far the simplest of the compactifications. It is obtained either by 'adding a point to the end of each geodesic', cf. [2], or equivalently by completing the stereographic image of $\mathfrak{a} \hookrightarrow \mathbb{S}(\mathfrak{a} \oplus \mathbb{R})$ as the closed upper hemisphere of S^n. This latter description immediately equips $\hat{\mathfrak{a}}$ with the structure of a smooth manifold with boundary. The monograph [17] contains an extended panegyric on the advantages of this space in the scattering analysis of the free Laplacian $\Delta_{\mathfrak{a}}$ and its (short range) perturbations. However, the lifts of the first order terms in Δ_{rad} to this space are not particularly simple, and this necessitates a slightly different approach. As a smooth manifold with corners, the compactification $\overline{\mathfrak{a}}$ is a slightly more complicated object, but it accomodates these first order terms very nicely. It is obtained essentially by requiring that the functions $e^{-\alpha}$ restricted to the positive Weyl chamber extend to smooth functions on the closure of C^+. However, although the principal part $\Delta_{\mathfrak{a}}$ lifts to a smooth b-operator on this space, it does not have a product structure near the corners, even asymptotically, and so its analysis here is still difficult. The space $\widetilde{\mathfrak{a}}$ is the smallest compactification for which there are smooth 'blowdown maps' to both $\hat{\mathfrak{a}}$ and $\overline{\mathfrak{a}}$, and it therefore has the property that *both* the principal part and the first order terms in Δ_{rad} lift nicely to this space. The precise sense in which we mean this will become apparent in the discussion below.

Through most of the ensuing discussion we tacitly assume that the root system Λ spans $\mathfrak{a}$. However, even if we start with a semisimple Lie algebra, where this is the case, we will always encounter situations in the overall induction on rank where $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}''$ and all roots vanish identically on the second summand. Therefore we must adapt all constructions and arguments to subsume this case too. Thus, to begin this generalization, the boundary of the radial compactification of $\mathfrak{a}$ is a sphere, inside of which sit the boundaries of the radial compactifications of the two summands as nonintersecting equatorial subspheres, and $\widehat{\mathfrak{a}}$ is the simplicial join of these subspheres, i.e.

$$\partial\,\widehat{\mathfrak{a}} = \partial\,\widehat{\mathfrak{a}'} \;\#\; \partial\,\widehat{\mathfrak{a}''}. \tag{2.5}$$

Of course, we regard $\partial\,\widehat{\mathfrak{a}}$ as a smooth (rather than a combinatorial) manifold.

2.4. The compactification $\overline{\mathfrak{a}}$.

The compactification $\overline{\mathfrak{a}}$ is known elsewhere in the symmetric space literature as the polyhedral or dual-cell compactification, see [6, Section 3.22-3.33]. It carries the natural structure of a polytope, i.e. is really a PL object, but for us it is only important that it is a smooth manifold with corners. Briefly, $\overline{\mathfrak{a}}$ is obtained by compactifying the positive Weyl chamber C^+ as a cube, $[0,1]^n$, to which the action of the Weyl group extends naturally; its translates by W fit together affinely to generate the entire polytope.

We now explain this more carefully. First fix an enumeration $\{\alpha_1, \ldots, \alpha_n\}$ of the set of positive simple roots Λ^+_{ind}. This is a basis for $\mathfrak{a}^*$, hence a maximal independent collection of linear coordinates on $\mathfrak{a}$. For any n-tuple $T = (T_1, \dots, T_n) \in \mathbb{R}^n$, there is an affine isomorphism

$$\mathcal{O}(T) := \bigcap_{j=1}^{n} \alpha_j^{-1}((T_j, +\infty)) \longrightarrow \prod_{j=1}^{n} (T_j, +\infty). \tag{2.6}$$

In particular, the positive Weyl chamber $C^+ = \mathcal{O}((0, \dots, 0))$ corresponds to the standard orthant $(\mathbb{R}^+)^n$. Now change variables, replacing α_j by $\tau_j := e^{-\alpha_j}$; the set $\mathcal{O}(T)$ is compactified by adjoining the faces where $\tau_j = 0$ and $\tau_j = e^{-T_j}$. Thus

$$\mathcal{O}(T) \subset \overline{\mathcal{O}(T)} \equiv \prod_{j=1}^{n} [T_j, \infty]_{\alpha_j} \cong \prod_{j=1}^{n} [0, e^{-T_j}]_{\tau_j}.$$

As already noted, $C^+ = \mathcal{O}(\vec{0})$, and so $\overline{C^+} = \overline{\mathcal{O}(\vec{0})}$. By definition, the smooth structure on these sets is the minimal one which agrees with the standard smooth structure on $\mathfrak{a}$ away from the outer boundaries and for which each τ_j is smooth. (Note, however, that $1/\alpha_j$ is *not* $\mathcal{C}^\infty$ on $\overline{\mathfrak{a}}$!)

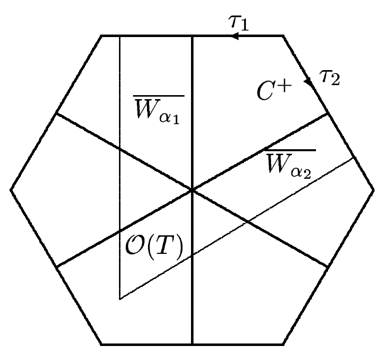

FIGURE 2. The compactification $\overline{\mathfrak{a}}$ for $M = \mathrm{SL}(3, \mathbb{R}) / \mathrm{SO}(3, \mathbb{R})$. The thick lines indicate the boundary faces and the Weyl chamber walls. The thin lines show the boundary of $\mathcal{O}(T)$ for $T_1 < 0$, $T_2 < 0$. The arrows indicate the coordinate axes τ_1 (i.e. $\tau_2 = 0$) and τ_2 (i.e. $\tau_1 = 0$) in the coordinate chart $\mathcal{O}(T)$.

Any other Weyl chamber is the positive chamber for a different set of indecomposable roots, and so may be compactified similarly. These compactifications fit together to cover all of $\overline{\mathfrak{a}}$. This shows that $\overline{\mathfrak{a}}$ is a topological cell, and provides it with a smooth structure away from these patching regions at the walls. To exhibit its structure as a smooth manifold with corners, observe that if all $T_j < 0$, then $\mathcal{O}(T) \supsetneq C^+$, and so these neighbourhoods cover the entire space $\mathfrak{a}$, and their completions patch together to cover all of $\overline{\mathfrak{a}}$ with open overlaps. Thus it suffices to show that for any $w \in W$, the restriction

$$w_T : w^{-1}(\mathcal{O}(T)) \cap \mathcal{O}(T) \to \mathcal{O}(T)$$

extends to a smooth map $\overline{w^{-1}(\mathcal{O}(T)) \cap \mathcal{O}(T)} \to \overline{\mathcal{O}(T)}$. For this, it is enough to prove that for any $\alpha_j \in \Lambda^+_{\mathrm{ind}}$, the function $w^* e^{-\alpha_j}$ extends smoothly to

$$\overline{w^{-1}(\mathcal{O}(T)) \cap \mathcal{O}(T)},$$

or equivalently, that $w^*\tau_j$ is smooth on this set. Now, $w^*\alpha_j$ is either in Λ^+ or Λ^-. In the former case, it decomposes as $\sum n_k\alpha_k$ where all n_k are nonnegative integers, and so

$$w^*\tau_j = \prod_k (e^{-\alpha_k})^{n_k} = \prod_k \tau_k^{n_k} \in \mathcal{C}^\infty(\overline{\mathcal{O}(T)}).$$

In the latter case, $w^*\alpha_j = -\sum n_k\alpha_k$, where the n_k are again all nonnegative. But the range of values of $w^*\alpha_j$ on $w^{-1}(\mathcal{O}(T))$ matches that of α_j on $\mathcal{O}(T)$, i.e. $w^*\alpha_j \geq T_j$ here. In addition, $\alpha_k \geq T_k$, on $\mathcal{O}(T)$. These inequalities imply that for each ℓ,

$$n_\ell\alpha_\ell = -\sum_{k\neq\ell} n_k\alpha_k - w^*\alpha_j \leq -\sum_{k\neq\ell} n_k T_k - T_j,$$

i.e. $n_\ell\alpha_\ell$ is bounded above on $w^{-1}(\mathcal{O}(T)) \cap \mathcal{O}(T)$. Hence either $n_\ell = 0$, or else α_ℓ is bounded above there. Writing $L = \{\ell : \; n_\ell \neq 0\}$,

$$w^* e^{-\alpha_j} = \prod_{\ell\in L} (e^{\alpha_\ell})^{n_\ell} = \prod_{\ell\in L} \tau_\ell^{-n_l},$$

which by the discussion above certainly extends smoothly to $\overline{w^{-1}(\mathcal{O}(T)) \cap \mathcal{O}(T)}$.

This proves that the transition maps are smooth, and hence that $\overline{\mathfrak{a}}$ has the structure of a smooth manifold with corners. This completes the construction.

Following the arguments of the previous paragraphs, we see that this 'bar compactification' construction commutes with taking products, i.e. if $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}''$, then

$$\overline{\mathfrak{a}} = \overline{\mathfrak{a}'} \times \overline{\mathfrak{a}''}. \tag{2.7}$$

Using this, we can directly adapt the construction to the reductive case, where the root system Λ vanishes identically on the second factor, once we have defined the appropriate compactification of an 'unadorned' Euclidean space $\mathfrak{b}$, with trivial root system. In this case, $\overline{\mathfrak{b}}$ is the 'logarithmic blow-down' of the radial compactification $\widehat{\mathfrak{b}}$. Namely, it is the smooth manifold with boundary such that $\overline{\mathfrak{b}}_{\log} = \widehat{\mathfrak{b}}$; in other words, if x is a smooth boundary defining function for $\widehat{\mathfrak{b}}$, then $\overline{\mathfrak{b}}$ is the same space as $\widehat{\mathfrak{b}}$, but with the smaller $\mathcal{C}^\infty$ structure, where by definition $e^{-1/x}$ is a boundary defining function. With this understanding, (2.7) defines the bar compactification even in the reductive case.

Let us now examine the lift of Δ_{rad} to $\overline{\mathfrak{a}}$. It suffices for now to restrict to any $\overline{\mathcal{O}(T)}$ where all $T_j > 0$ (to avoid the Weyl chamber walls). We can study the form of this operator near $\partial\,\overline{\mathfrak{a}}$ by changing variables from $\{\alpha_1, \ldots, \alpha_n\}$ to $\{\tau_1, \ldots, \tau_n\}$. We have $\partial_{\alpha_j} = -\tau_j\partial_{\tau_j}$, and these latter vector fields generate $\mathcal{V}_b(\overline{\mathfrak{a}})$, the space of smooth b vector fields on $\overline{\mathfrak{a}}$; by definition $\mathcal{V}_b$ consists of all smooth vector fields on $\overline{\mathfrak{a}}$ which are unconstrained in the interior but lie tangent to all boundaries. Thus, all translation-invariant vector fields on $\mathfrak{a}$ lift to elements of $\mathcal{V}_b(\overline{\mathfrak{a}})$, and indeed the latter is generated by the lifts of these vector fields over $\mathcal{C}^\infty(\overline{\mathfrak{a}})$. Hence, all translation-invariant differential operators on $\mathfrak{a}$ lift to elements of $\mathrm{Diff}^*_{\mathrm{b}}(\overline{\mathfrak{a}})$, the space of operators which can be written locally as finite sums of elements of $\mathcal{V}_b(\overline{\mathfrak{a}})$.

In particular, the principal part $\Delta_{\mathfrak{a}}$ is transformed to an elliptic, constant coefficient combination of these basic b vector fields. In addition, $\coth\alpha - 1$ is a $\mathcal{C}^\infty$ function on $\mathfrak{a}$ away from the Weyl chamber walls. Indeed, $\coth\alpha - 1 = 2e^{-2\alpha}/(1-e^{-2\alpha})$, and so for $\alpha = \sum n_j \alpha_j \in \Lambda^+$, we have

$$\coth\alpha - 1 = \frac{\exp(-2\sum_{j=1}^n n_j\alpha_j)}{1-\exp(-2\sum_{j=1}^n n_j\alpha_j)} = \frac{\prod_{j=1}^n \tau_j^{2n_j}}{1-\prod_{j=1}^n \tau_j^{2n_j}},$$

which is certainly a $\mathcal{C}^\infty$ function of the τ_j if $\tau_k < 1$ for all k. Since

$$H_\alpha = \sum_{j=1}^n n_j \partial_{\alpha_j} = \sum_{j=1}^n n_j(-\tau_j\partial_{\tau_j})$$

is a translation-invariant vector field on $\mathfrak{a}$, we deduce that away from the Weyl chamber walls, Δ_{rad} is indeed an elliptic element of $\mathrm{Diff}^2_{\mathrm{b}}(\overline{\mathfrak{a}})$.

This may lead one to conclude that, except possibly having to deal with some technicalities along the walls (which could be eliminated by working on the analogous compactification $\overline{M}$ of M which we define later), $\mathrm{Diff}^*_{\mathrm{b}}(\overline{\mathfrak{a}})$ is the appropriate setting to analyze Δ_{rad}. However, this is not the case since the techniques of the so-called b-calculus on manifolds with corners only applies for operators which are asymptotically of product type near the corners. This is unfortunately false for Δ_{rad}, ultimately because the α_j are not orthogonal, but we now explain this more carefully.

The roots α_j are the linear coordinates for the dual basis $K_1, \dots, K_n$ of $\mathfrak{a}$ associated to Λ^+_{ind} (by $\alpha_i(K_j) = \delta_{ij}$ for all i, j). If $e_1, \dots, e_n$ is any orthonormal basis for $\mathfrak{a}$, then any vector $v \in \mathfrak{a}$ can be expressed in terms of either basis:

$$v = \sum_{j=1}^n y_j e_j = \sum_{\ell=1}^n x_\ell K_\ell.$$

Letting $\mathcal{K}$ be the matrix with columns $K_1, \dots, K_n$, then $y = \mathcal{K}x$, and so if $\mathcal{K}^{-1} = \mathcal{H} = (H_{rs})$, then we have

$$\Delta_{\mathfrak{a}} = \sum_{i=1}^n \frac{\partial^2}{\partial y_i^2} = \sum_{i,p,q=1}^n \frac{\partial x_p}{\partial y_i}\frac{\partial x_q}{\partial y_i}\frac{\partial^2}{\partial x_p \partial x_q} = \sum_{i,p,q=1}^n H_{pi}H_{qi}\frac{\partial^2}{\partial x_p\partial x_q}.$$

Next, associated to each α_j is the metrically dual vector H_j, i.e. $\alpha_j(w) = \langle H_j, w\rangle$ for all $w \in \mathfrak{a}$. Then $\alpha_j(K_i) = \delta_{ij} = \langle H_j, K_i\rangle$, which means that the matrix $\mathcal{H} = \mathcal{K}^{-1}$ appearing above has columns equal to the vectors $H_1, \dots, H_n$. We have thus shown that

$$\Delta_{\mathfrak{a}} = \sum_{p,q=1}^n \gamma_{pq}\frac{\partial^2}{\partial x_p\partial x_q}, \tag{2.8}$$

where $\Gamma = (\gamma_{pq}) = \mathcal{H}\mathcal{H}^t$. Finally, in terms of the coordinates $\tau_j = e^{-\alpha_j}$, we have

$$\Delta_{\mathfrak{a}} = \sum_{p,q=1}^n \gamma_{pq}(\tau_p\partial_{\tau_p})(\tau_q\partial_{\tau_q}). \tag{2.9}$$

However, the matrix Γ is usually not diagonal, i.e. $\Delta_{\mathfrak{a}}$ is not 'product-type'.

2.5. **The compactification $\widetilde{\mathfrak{a}}$.** We now describe the final, dominating, compactification $\widetilde{\mathfrak{a}}$. This is adapted from a compactification used in more general many-body settings, as initially defined by the second author and employed in [24]. We first present this from the general point of view, not using the roots or the Weyl group action, but only the existence of a finite lattice $\mathcal{S}$ of subspaces of the ambient space

$\mathfrak{a} = \mathbb{R}^n$. This first construction of $\widetilde{\mathfrak{a}}$ does not pass through $\overline{\mathfrak{a}}$ as an intermediate space, but at the end of the section we discuss the relationship between the two spaces $\overline{\mathfrak{a}}$ and $\widetilde{\mathfrak{a}}$ and present a different construction of the latter space which does pass through the former.

Let $\mathcal{S}$ be the collection of all intersections of Weyl chamber walls W_α (as well as the 'empty intersection' $\mathfrak{a}$); this is a lattice, since it is closed under intersections and contains both $\{0\}$ and $\mathfrak{a}$. We index this collection by a set I, so $\mathcal{S} = \{S_b : b \in I\}$; in particular, we suppose that $\{0, *\} \subset I$, where $S_0 = \mathfrak{a}$ and $S_* = \{0\}$. Finally, for any $S_b \in \mathcal{S}$, write S^b for the orthocomplement $S_b^\perp$.

Now let us proceed with the construction. In the first step we pass to the radial (or geodesic) compactification $\hat{\mathfrak{a}}$, which is obtained by (hemispherical) stereographic projection, or alternatively, by compactifying each ray $\mathfrak{r} \cong [0, \infty)$ emanating from a fixed basepoint $o \in \mathfrak{a}$ as a closed interval $[0, \infty]$. As described earlier, there is a natural topology and differential structure which makes $\hat{\mathfrak{a}}$ into a smooth manifold with boundary.

Next, let C_b be the boundary of the closure of S_b in $\hat{\mathfrak{a}}$; this is a great sphere of dimension $\dim S_b - 1$. The collection of all such great spheres $\mathcal{C} = \{C_b : b \in I\}$ is again a lattice. The singular and regular parts of C_b are defined by

$$C_{b,\mathrm{sing}} = \bigcup\{C_c : C_c \subsetneq C_b\}, \qquad C_{b,\mathrm{reg}} = C_b \setminus C_{b,\mathrm{sing}},$$

and the singular and regular parts of S_b are defined analogously. The space $\widetilde{\mathfrak{a}}$ is obtained by blowing up the collection $\mathcal{C}$ inductively, in order of increasing dimension, as follows. $\mathcal{S}$ is a union of subcollections $\mathcal{S}_j$, where $\dim S = j$ for any $S \in \mathcal{S}_j$. We first blow up the set of points C_b corresponding to $S_b \in \mathcal{S}_1$ to obtain a space $\hat{\mathfrak{a}}^{(1)}$. Next, define the collection $\mathcal{C}^{(1)}$ of submanifolds with boundary obtained by lifting the regular parts $C_{b,\mathrm{reg}}$ of each of the remaining sets C_b and taking their closures in $\hat{\mathfrak{a}}^{(1)}$. This is again a lattice, but the minimal dimension of its elements is now 1, corresponding to elements $S_b \in \mathcal{S}_2$; furthermore, these 1-dimensional submanifolds with boundary are disjoint. We blow these up to form a space $\hat{\mathfrak{a}}^{(2)}$. Continue this process, obtaining a sequence of spaces $\hat{\mathfrak{a}}^{(\ell)}$ and lattices of submanifolds $\mathcal{C}^{(\ell)}$ with components of dimension greater than or equal to ℓ, and with all ℓ-dimensional components disjoint submanifolds with corners. We obtain after n steps the space $\widetilde{\mathfrak{a}} := \hat{\mathfrak{a}}^{(n)}$. This compactification is a smooth manifold with corners, and is equipped with a smooth blow-down map $\beta : \widetilde{\mathfrak{a}} \to \hat{\mathfrak{a}}$.

Notice that the indices $b \in I \setminus \{*\}$ are in bijective correspondence with the codimension one boundary faces of $\widetilde{\mathfrak{a}}$, and also with the boundary faces of arbitrary codimension of $\overline{\mathfrak{a}}$. Thus associated to any C_b is the (possibly disconnected) boundary hypersurface $\tilde{F}_b$ of $\widetilde{\mathfrak{a}}$, and higher codimensional boundary face $\overline{F}_b$ of $\overline{\mathfrak{a}}$. This suggests the alternate definition of $\widetilde{\mathfrak{a}}$ as the logarithmic total boundary blow-up of $\overline{\mathfrak{a}}$. More specifically, first replace each boundary defining function τ_j of $\overline{\mathfrak{a}}$ by $\overline{\tau}_j = -1/\log \tau_j$; then blow up the corners of $\overline{\mathfrak{a}}$ inductively, in order of increasing dimension. This is essentially dual to the previous construction. In fact, the face $\tilde{F}_0$, corresponding to $S_0 = \mathfrak{a}$ and $C_0 = S^{n-1}$, is the face obtained in this alternate definition by blowing up the highest codimension corners of $\overline{\mathfrak{a}}$. Similarly, the faces $\tilde{F}_j$ created at the first stage in the first definition of $\widetilde{\mathfrak{a}}$ by blowing up the one dimensional elements C_1 correspond to the hypersurface faces of $\overline{\mathfrak{a}}$. All other faces of $\widetilde{\mathfrak{a}}$ correspond to the various intermediate codimension corners in $\overline{\mathfrak{a}}$. In any case, blowups of the boundary hypersurfaces of $\overline{\mathfrak{a}}$ occur as boundary hypersurfaces

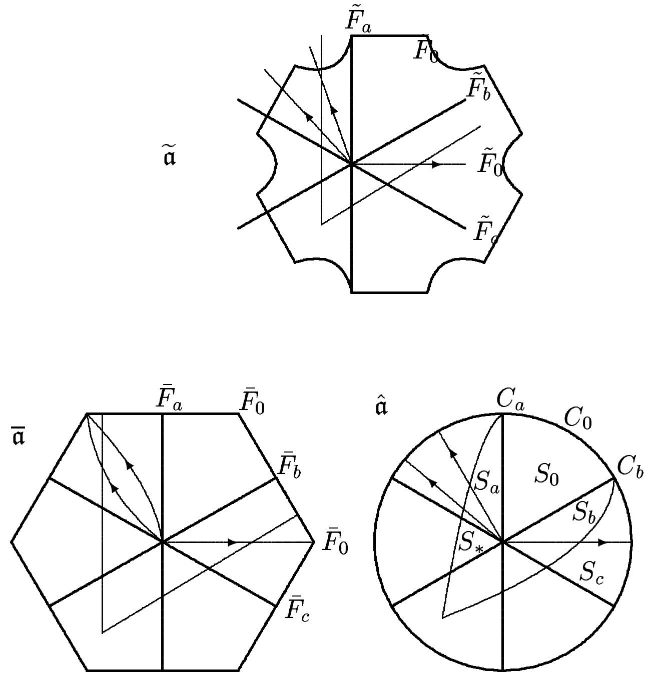

FIGURE 3. Representation of the compactifications $\overline{\mathfrak{a}}$, $\hat{\mathfrak{a}}$ and $\widetilde{\mathfrak{a}}$ for $M = \mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3,\mathbb{R})$. The thick lines indicate the boundary faces and the Weyl chamber walls. The thin lines without arrows show the boundary of the closure of $\mathcal{O}(T)$, for $T_1 < 0$, $T_2 < 0$, in the various compactifications. The thin lines with arrows are geodesic rays emanating from 0; in particular they bound *conic* regions. Geodesic rays in a single Weyl chamber in $\overline{\mathfrak{a}}$ hit the same point on $\partial\overline{\mathfrak{a}}$, whereas in $\hat{\mathfrak{a}}$, the boundary lines of $\mathcal{O}(T)$ hit C_a and C_b for any T.

of $\widetilde{\mathfrak{a}}$, but that there are many other boundary hypersurfaces of this latter space, or in other words, $\widetilde{\mathfrak{a}}$ distinguishes more directions of approach to infinity. The replacement of each defining function by its logarithm here reflects the fact that in the ball model of hyperbolic space, for example, the defining function x is essentially $\exp(-\mathrm{dist})$, while in the stereographic compactification of Euclidean space, the defining function x is $1/\mathrm{dist}$. We refer to §6 of [14] for an extensive discussion of the role of smooth defining functions in compactification theory.

The behaviour of this 'tilde compactification' with respect to taking products is a bit more complicated than for the bar compactification. First of all, if the root system of $\mathfrak{a}$ is trivial, i.e. $\mathfrak{a}$ is an unadorned Euclidean space, then $\widetilde{\mathfrak{a}} = \hat{\mathfrak{a}} = \overline{\mathfrak{a}}_{\log}$. Secondly, if $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}''$, then $\widetilde{\mathfrak{a}}$ is obtained by blowing up the closed ball $\hat{\mathfrak{a}}$ along the collection of boundary submanifolds $\mathcal{C} = \{C_a\} = \{\partial S_a\}$, where each S_a is of the form $S_b' \times S_c''$ (including, of course, the cases $S_b' = \{0\}$ or $S_c'' = \{0\}$). Hence C_a is either the simplicial join $C_b' \# C_c''$ (regarded as a smooth great sphere in $\partial\hat{\mathfrak{a}}$) or else $C_b' \times \{0\}$ or $\{0\} \times C_c''$; in particular, if all roots vanish on $\mathfrak{a}''$, then each C_a equals either $C_b' \# \partial\hat{\mathfrak{a}}''$ or $C_b' \times \{0\}$. Of course, we can also obtain $\widetilde{\mathfrak{a}}$ as the total boundary blowup of $\overline{\mathfrak{a}}$, i.e. as

$$\widetilde{\mathfrak{a}} = \left[(\overline{\mathfrak{a}})_{\log}; \overline{\mathcal{F}}\right] = \left[\left(\overline{\mathfrak{a}}' \times \overline{\mathfrak{a}}''\right)_{\log}; \overline{\mathcal{F}}\right] = \left[\left(\overline{\mathfrak{a}}'\right)_{\log} \times \left(\overline{\mathfrak{a}}''\right)_{\log}; \overline{\mathcal{F}}\right], \tag{2.10}$$

where $\overline{\mathcal{F}}$ is the collection of boundary faces of all codimension in $\overline{\mathfrak{a}}$. If all roots vanish on $\mathfrak{a}''$, then

$$\widetilde{\mathfrak{a}} = \left[\left(\overline{\mathfrak{a}}'\right)_{\log} \times \widehat{\mathfrak{a}''}; \left(\overline{\mathcal{F}'} \times \widehat{\mathfrak{a}''}\right) \cup \left(\left(\overline{\mathfrak{a}}'\right)_{\log} \times \partial \widehat{\mathfrak{a}''}\right)\right]. \tag{2.11}$$

2.6. **Compactifications of the full symmetric space.** Before continuing with the more detailed description of Δ_{rad} on $\widetilde{\mathfrak{a}}$, we follow the train of thought from the past two subsections and define the compactifications $\overline{M}$ and $\widetilde{M}$ of the full symmetric space M, corresponding to $\overline{\mathfrak{a}}$ and $\widetilde{\mathfrak{a}}$, respectively. Their role in this paper is only minor since our emphasis is on the radial Laplacian. Nevertheless, many properties of the operator Δ_{rad}, which has nonsmooth coefficients on $\mathfrak{a}$, are proved by appealing to its lift to M, which is just the operator Δ, and which does have smooth coefficients; we also consider lifts of Δ_{rad} to certain spaces intermediate between between the various compactifications of M and $\mathfrak{a}$.

As we have seen in §2.1, the Cartan decomposition $G = K\overline{\mathcal{C}^+}K$ states that any $g \in G$ has a decomposition $k_1 \cdot a \cdot k_2$, where $k_1, k_2 \in K$ and $a = \exp(H)$, $H = H(g) \in \overline{\mathcal{C}^+}$, and with this normalization, a is *unique*. Moreover, if $p \in M = G/K$ has $H(p) \in \mathcal{C}^+$ then K^p, the subgroup of K that fixes p, is discrete; the set of such p is open and dense in M and is diffeomorphic to $(K/K^{p_0}) \times \mathcal{C}^+$ (for any $p_0 \in \mathcal{C}^+$).

As discussed in §2.6, each (open) face S_b^+ of the closed positive Weyl chamber $\overline{\mathcal{C}^+}$ in $\mathfrak{a}$ is an open set in a unique S_b, $b \in I$, and we index the set of all such faces S_b^+ by a subset $I^+ \subset I$.

If $p \in \exp(S_{b,\mathrm{reg}} \cap \overline{\mathcal{C}^+})$, $b \in I^+$, let Λ_b be the set of roots vanishing at p. Since $S_b \subset \mathfrak{a} \subset \mathfrak{g}_0$, there is an orthogonal splitting $\mathfrak{g}_0 = S_b \oplus \mathfrak{g}_0^b$, and we then define

$$\mathfrak{g}^b = \mathfrak{g}_0^b \oplus \sum_{\alpha \in \Lambda_b} \mathfrak{g}_\alpha, \qquad \text{and} \quad \mathfrak{p}^b = \mathfrak{p} \cap \mathfrak{g}^b,$$

cf. [2, Section 2.20]. This is the Lie algebra of a Lie subgroup $G^b \subset G$, which contains the isotropy group of p in K. Denoting this latter group by K^b, and its Lie algebra by $\mathfrak{k}^b$, then $\mathfrak{g}^b = \mathfrak{k}^b \oplus \mathfrak{p}^b$. There is a corresponding symmetric space $\Sigma^b = G^b/K^b$, which is identified with $\exp(\mathfrak{p}^b)$. Now, the image N of a neighbourhood of $(S_{b,\mathrm{reg}} \cap \overline{\mathcal{C}^+}) \times \{0\}$ in $(S_{b,\mathrm{reg}} \cap \overline{\mathcal{C}^+}) \times \mathfrak{p}^b$ under exp is a submanifold of M, with p lying on it, and the K-action is transversal to N at p. Thus, a neighbourhood of the K-orbit of p is diffeomorphic to the K-orbit of the K^b-class of $(H(p), e, o)$, where e is the identity element in K and o the identity coset in Σ^b, in

$$S_b \times (K \times \Sigma^b)/K^b, \qquad \text{where} \quad k_1 \cdot (k, \sigma) = (k k_1^{-1}, k_1 \cdot \sigma) \quad \text{for any } k_1 \in K^b.$$

We can let p vary in $\exp(S_{b,\mathrm{reg}} \cap \overline{\mathcal{C}^+})$, and deduce that a neighborhood of the K-orbit of $\exp(S_{b,\mathrm{reg}} \cap \overline{\mathcal{C}^+})$ is diffeomorphic to the K-orbit of the K^b-class of $(S_{b,\mathrm{reg}} \cap \overline{\mathcal{C}^+}) \times \{e\} \times \{o\}$. Reinterpreted, this says that the K-orbit of a neighbourhood of $\exp(S_{b,\mathrm{reg}} \cap \overline{\mathcal{C}^+})$ in M is a $\mathcal{C}^\infty$ bundle over $K/K^b \times \exp(S_{b,\mathrm{reg}} \cap \overline{\mathcal{C}^+})$ with fibre (a neighbourhood of the origin in) Σ^b.

In fact, this argument shows more. Consider the action of $\mathbb{R}^+$ by dilations on $\mathfrak{p}$: $\mathbb{R}^+ \times \mathfrak{p} \ni (t, z) \mapsto tz \in \mathfrak{p}$. A set is called conic if it is invariant under the

$\mathbb{R}^+$-action. As remarked before, this $\mathbb{R}^+$-action on $\mathfrak{p}$ is identified with dilations along the geodesic rays through o via the exponential map. Now, $k \cdot \exp(tX) = \exp(\mathrm{Ad}(k)(tX)) = \exp(t\,\mathrm{Ad}(k)X)$ for $k \in K$, $X \in \mathfrak{p}$, $t \in \mathbb{R}^+$. Thus, under the identification of a neighbourhood of p as above with a neighbourhood of $(e, o, 0) \in (K/K^b) \times \Sigma^b \times S_b$, the $\mathbb{R}^+$-action is $(t, kK^b, q, x) \mapsto (t, kK^b, tq, tx)$, at first for t near 1. Thus, we can extend the identification to a conic neighbourhood of the $\mathbb{R}^+$-orbit of p via the dilation. Letting p vary in a bounded set, we deduce that there is a conic neighbourhood U_b of $S_{b,\mathrm{reg}} \cap \overline{C^+}$ in $\mathfrak{a}$ such that $K \cdot \exp(U_b)$ can be identified with a $\mathcal{C}^\infty$ bundle over $K/K^b \times \exp(S_{b,\mathrm{reg}} \cap \overline{C^+})$ with fibre (a neighbourhood of the origin in) Σ^b. We let Φ_b be this identification.

If $p \in \exp(S_{c,\mathrm{reg}} \cap \overline{C^+}) \cap \exp(U_b)$, then $S_b \subset S_c$ and $p \in \exp(U_c)$ as well, so there are two identifications of a conic neighbourhood of p: one as a subset of $(K/K^b) \times \Sigma^b \times S_b$, and the other as a subset of $(K/K^c) \times \Sigma^c \times S_c$. Since $K^c \subset K^b$, we have $K/K^b \subset K/K^c$ and $\Sigma^c \subset \Sigma^b$. The map between these two identifications is thus a diffeomorphism, and it commutes with the $\mathbb{R}^+$-action.

We can now define $\overline{M}$; this is called the dual cell compactification of M, see [6, Section 3.40], where it is defined as a topological space with a G-action. Our construction proceeds by partially compactifying part of the regions described in the preceeding paragraphs. Thus, we fix a K^b-invariant bounded neighbourhood $\mathcal{O}_b$ of o in each symmetric space Σ^b; this has a W^b-invariant bounded intersection O_b with S^b. Let V_b be an open subset of $S_{b,\mathrm{reg}}$ such that $S_{b,\mathrm{reg}} \setminus V_b$ is bounded and $V_b \times O_b \subset U_b$. Such a subset exists since U_b is a conic neighbourhood of $S_{b,\mathrm{reg}} \cap \overline{C^+}$. Then, by the preceeding discussion, $K \cdot \exp(V_b \times O_b)$ is a $\mathcal{C}^\infty$ bundle over $(K/K^b) \times V_b$ with fiber $\mathcal{O}_b$. We partially compactify the base of this bundle as $(K/K_b) \times \overline{V_b}$, where $\overline{V_b}$ is the closure of V_b in $\overline{S_{b,\mathrm{reg}}}$, the regular part of the bar-compactification of S_b.

If now c is such that $S_b \subset S_c$, then we have seen that on $K \cdot \exp((V_b \times O_b) \cap (V_c \cap O_c))$ the transition maps between the identifications of the respective bundles is a diffeomorphism. It is now immediate that the same is true in these partial compactifications since this amounts to showing that the identification map on the subset $(V_b \times O_b) \cap (V_c \times O_c)$ of $\mathfrak{a}$ extends to be smooth on $(\overline{V_b} \times O_b) \cap (\overline{V_c} \times O_c)$, which is immediate from the definition of $\overline{\mathfrak{a}}$.

We can thus define $\overline{M}$ as the disjoint union of the $\mathcal{O}_b$-bundles over $(K/K_b) \times \overline{V_b}$, $b \in I^+$, modulo the equivalence relation corresponding to this identification. Then $\overline{M}$ is a manifold with corners – the corners arise from the $\overline{V_b}$, i.e. from the compactification of the flat.

Even though we have remained in a bounded neighbourhood of o in each symmetric space Σ^b to avoid a recursive definition of the compactifications, it is now immediate that the boundary faces $\overline{\mathcal{F}_b}$, $b \in I^+$, of $\overline{M}$ are $\mathcal{C}^\infty$ bundles over K/K_b with fiber $\overline{\Sigma^b}$ (the bar-compactification of Σ^b). Indeed, this simply relies on considering the closure of the conic set $K \cdot \exp(U_b)$ in $\overline{M}$. Note, however, that this closure does *not* include a neighbourhood of $\overline{\mathcal{F}_b}$. Indeed, the issue is that the closure of U_b in $\overline{\mathfrak{a}}$ does not include a neighbourhood of the face $\overline{F_b}$, though it *does* contain a neighbourhood of the open face F_b.

This procedure may be modified easily for the construction of $\widetilde{M}$. Indeed, in each step we simply replace $\overline{V_b}$ by $\widetilde{V_b}$, the closure of V_b in $\widetilde{S_{b,\mathrm{reg}}}$, the regular part of the bar-compactification of S_b. By the naturality of all the steps, it is clear that we could also define $\widetilde{M}$ as the logarithmic total boundary blow-up of $\overline{M}$.

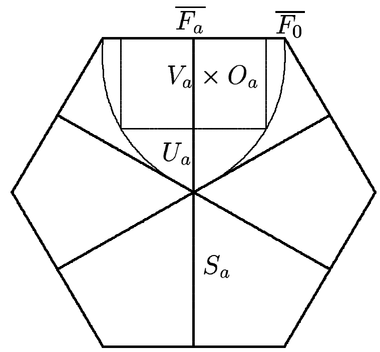

FIGURE 4. Subsets of $\overline{\mathfrak{a}}$ used in the construction of $\overline{M}$ for $M = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$. The thick lines indicate the boundary faces and the Weyl chamber walls. The rectangular thin lines show the boundary of $V_a \times O_a$. The curved ones indicate the boundary of U_a; they are in particular geodesic rays from o. The corresponding subsets for $b = 0$ are $U_0 = \mathcal{C}^+$, the positive Weyl chamber, $O_0 = \{o\}$ and $V_0 = \mathcal{C}^+$. Thus, the 0-chart covers a neighbourhood of the corner, $\overline{F_0}$.

We recall that as a topological space, it is described in [6] as the smallest compactification that dominates both $\overline{M}$ and the geodesic (or conic) compactification $\hat{M}$. Note that the latter does not have a natural smooth structure: if it is defined by compactifying $\mathfrak{p}$ radially and using the exponential map, the smooth structure depends on the choice of the base point o. It is shown in [6, Theorem 8.21] that, as a topological space, $\widetilde{M}$ is the Martin compactification of M.

Remark 2.2. Although we have defined $\overline{M}$ and $\widetilde{M}$, we never actually use them in this paper. Rather, since we are working with K-invariant functions and operators, the only reason to leave $\mathfrak{a}$ (or $\overline{\mathfrak{a}}$ and $\widetilde{\mathfrak{a}}$) is to make the differential operators have smooth coefficients. For this purpose, the K/K_b factor can be ignored, and we may work instead on $V_b \times \mathcal{O}_b$, etc, which is exactly what we do in § 4. However, it is nice to know that there is a compactification of M in the background, rather than just an ad hoc collection of product spaces!

2.7. **The lift of $\Delta_{\mathfrak{a}}$ to $\widetilde{\mathfrak{a}}$.** In the remaining subsections of §2 we shall be examining the structure of Δ_{rad} on $\widetilde{\mathfrak{a}}$ in some detail, focusing specifically on its behaviour at and near the boundary. This involves several steps. In this subsection we study the lift of the flat Laplacian $\Delta_{\mathfrak{a}}$, and vindicate our earlier claim that this operator attains a product-type structure near the corners of $\widetilde{\mathfrak{a}}$. The results of this section are not used elsewhere in the paper.

Recall the expression (2.9), which exhibits Δ_{rad} as an elliptic b-operator on $\overline{\mathfrak{a}}$. We now introduce a singular change of variables on $\overline{\mathfrak{a}}$. Using multi-index notation, set

$$\sigma = \tau^\theta, \qquad \text{i.e.} \quad \sigma_i = \tau_1^{\theta_{i1}} \dots \tau_n^{\theta_{in}},$$

where $\Theta = (\theta_{ij})$ is some n-by-n matrix to be determined. We calculate

$$\tau_s \partial_{\tau_s} = \sum_{r=1}^{n} \theta_{rs} \sigma_r \partial_{\sigma_r},$$

and so

$$\Delta_{\mathfrak{a}} = \sum \gamma_{pq} \theta_{ip} \theta_{jq} (\sigma_i \partial_{\sigma_i})(\sigma_j \partial_{\sigma_j}) = \sum \nu_{ij} (\sigma_i \partial_{\sigma_i})(\sigma_j \partial_{\sigma_j}),$$

where $N = (\nu_{ij}) = \Theta \Gamma \Theta^t$. We wish to choose Θ so that N is diagonal. We intend to study $\Delta_{\mathfrak{a}}$ (and Δ_{rad}) near the closure of some face F, which we label for simplicity as $\tau_1 = 0$; the ordering of the other faces is then arbitrary. Relative to this ordering, since Γ is positive definite, there is a factorization $\Gamma = LDU$, where L and U are lower and upper triangular, respectively, and D is diagonal. Since this factorization is unique, and $\Gamma = \Gamma^t$, we must have $U = L^t$. Hence if we define $\Theta = L^{-1}$, which is also lower triangular, then $L^{-1}\Gamma(L^{-1})^t = N$ is the diagonal matrix D appearing in the decomposition, as desired. Somewhat more explicitly, this coordinate change has the form

$$\sigma_1 = \tau_1, \;\; \sigma_2 = \tau_1^{\theta_{21}} \tau_2, \ldots \sigma_n = \tau_1^{\theta_{n1}} \cdots \tau_{n-1}^{\theta_{n\,n-1}} \tau_n.$$

We have now shown that $\Delta_{\mathfrak{a}}$ may be transformed to diagonal form near any corner of $\overline{\mathfrak{a}}$, but at the expense of using a singular coordinate change.

The other key step is to show that this singular coordinate change lifts to a smooth (local) diffeomorphism of $\widetilde{\mathfrak{a}}$. Recall that this latter space is obtained by first introducing the logarithmic change of variables $\overline{\tau}_i = -1/\log \tau_i$, and then blowing up the corners in order of increasing dimension. Defining $\overline{\sigma}_i = -1/\log \sigma_i$, then

$$\frac{1}{\overline{\sigma}_1} = \frac{1}{\overline{\tau}_1}, \quad \ldots \quad , \quad \frac{1}{\overline{\sigma}_j} = \frac{\theta_{j1}}{\overline{\tau}_1} + \ldots + \frac{\theta_{j\,j-1}}{\overline{\tau}_{j-1}} + \frac{1}{\overline{\tau}_j}, \quad \ldots$$

These formulæ represent the lift of this map acting between $(\overline{\mathfrak{a}})_{\log}$, but it is still not smooth. The passage to the total boundary blowup fixes this: to this end, first note that each $\overline{\sigma}_j$ is homogeneous of degree 1 in the $\overline{\tau}_i$, and so if we introduce polar coordinates $\overline{\tau} = r\omega$, $\overline{\sigma} = r'\phi$ near $\overline{\tau} = \overline{\sigma} = 0$, then we can identify the radial variables, $r = r'$. For simplicity, we examine this near the codimension 2 corners of the blowup, i.e. near where exactly one of the ω_i vanish, and away from the higher codimension corners where two or more of these angular variables equal zero. Thus suppose we are working near $\omega_j = 0$. For every k we have

$$\frac{1}{\phi_k} = \frac{\theta_{k1}}{\omega_1} + \ldots + \frac{\theta_{k\,k-1}}{\omega_{k-1}} + \frac{1}{\omega_k}. \tag{2.12}$$

Thus if $k < j$ then ϕ_k is obviously a smooth function of ω since all terms here are nonvanishing (note that the whole right hand side cannot vanish, since otherwise we would reach the incorrect conclusion that $\overline{\sigma}_k$ itself would be undefined). Next, if $k = j$, then we can rewrite (2.12) as

$$\phi_j = \frac{\omega_j}{\theta_{j1}\frac{\omega_j}{\omega_1} + \ldots + \theta_{j\,j-1}\frac{\omega_j}{\omega_{j-1}} + 1},$$

which again is certainly smooth. Finally, if $k > j$, then

$$\phi_k = \frac{\omega_k}{\theta_{k1}\frac{\omega_j}{\omega_1} + \ldots + \theta_{kj} + \ldots + \frac{\omega_j}{\omega_k}};$$

if $\theta_{kj} \neq 0$, then this is smooth near $\omega_k = 0$, while if $\theta_{jk} = 0$, then ϕ_k is independent of ω_j, hence again is smooth. The argument near the higher codimension corners is similar.

2.8. **Subsystems.** We now consider the restrictions of Δ_{rad} to the codimension one boundary faces of $\widetilde{\mathfrak{a}}$; our goal is to show that each such restriction is essentially the radial Laplacian on some lower rank symmetric space. To this end, we examine the geometry of $\partial\widetilde{\mathfrak{a}}$ more closely.

2.8.1. *Geometric and algebraic subsystems.* Any point $p \in \partial\hat{\mathfrak{a}}$ belongs to a unique $C_{b,\text{reg}}$ for some $b \in I$. Note that $C_c \cap C_{b,\text{reg}} \neq \emptyset$ only when $C_c \supset C_b$, or equivalently when $S_c \supset S_b$. Thus, in particular, for any root α, the wall W_α equals S_c for some $c \in I$, and the corresponding C_c intersects $C_{b,\text{reg}}$ only when $W_\alpha \supset S_b$. Thus p has a neighbourhood $\mathcal{U}$ in $\hat{\mathfrak{a}}$ such that $\mathcal{U} \cap W_\alpha \neq \emptyset$ only when $S_b \subset W_\alpha$.

Next, the boundary hypersurfaces F of $\widetilde{\mathfrak{a}}$ are in one-to-one correspondence with the indices $b \in I \setminus \{*\}$, where F_b is the front face created by blowing up $C_{b,\text{reg}}$. The interior of each F_b has a (trivial) fibration induced by the blow-down map β, with base $C_{b,\text{reg}}$ and fibre the orthocomplement S^b. We remark that this extends to a fibration of the closed face F_b, with fibre $\widetilde{S^b}$, the compactification of S^b obtained analogously to $\widetilde{\mathfrak{a}}$ by regarding S^b as a flat in the lower rank symmetric space Σ^b, and base the closure of the lift of $C_{b,\text{reg}}$ in the partially blown up space $\hat{\mathfrak{a}}^{(\ell)}$, $\ell = \dim C_b$. The base can also be identified with the lift of C_b to $\widetilde{S_b} = [\widehat{S_b}; \{C_c : \ C_c \subsetneq C_b\}]$. Indeed, this is description is *identical* to the geometry of compactifications in N-body scattering; see [24, pp. 339-340] for a very detailed discussion of the latter.

Translating by an element of the Weyl group, we can suppose that $p \in \overline{C^+}$. Let us then say that a root α is positive, negative, or zero at p if α has this property on the ray in $\mathfrak{a}$ corresponding to p. In particular, α vanishes at p (and at every other $q \in C_{b,\text{reg}}$ as well) if and only if $W_\alpha \supset S_b$.

Let Λ_b denote the subset of all roots α which vanish on S_b. We have identifications

$$\{\gamma \in \mathfrak{a}^* : \gamma = 0 \text{ on } S_b\} \cong (\mathfrak{a}/S_b)^* \cong (S^b)^*;$$

the first of these is tautological, while the second uses the metric, but both are isometries. Hence we can also regard $\Lambda_b \subset (S^b)^*$, with the same inner product relations as in $\mathfrak{a}^*$, and clearly this is a spanning set of covectors. In addition, $\alpha \in \Lambda_b$ if and only if $W_\alpha^\perp \subset S^b$, or equivalently $H_\alpha \in S^b$. It is now easy to check that Λ_b satisfies all the axioms of a reduced root system on $\operatorname{span}(\Lambda_b) \subset (S^b)^*$, cf. [10, Section 9.2]. We define $\Lambda_b^+ = \Lambda_b \cap \Lambda^+$.

In conclusion, we have shown that for each $b \in I \setminus \{*\}$, $\mathfrak{a} = S_b \oplus S^b$, where the latter summand is the Cartan subspace for some symmetric space of rank less than n; furthermore, the face F_b is the product of the base space, which is a compactification of $C_{b,\text{reg}}$, and the radial compactification of the vector space S^b. There is a more familiar geometric version of this statement. Fix $p \in C_{b,\text{reg}}$ and let γ be the geodesic in M which is the exponential of the ray corresponding to p. We say that another geodesic γ' is parallel to γ if the two geodesics stay a bounded distance from one another in both directions. Following [2], we define $F(\gamma)$ to be the union of all geodesics parallel to γ. This is a totally geodesic submanifold in M, and it always admits a Riemannian product decomposition $\mathbb{R}^k \times F_s(\gamma)$, where the second factor is a symmetric space of rank strictly less than n. The correspondence is that the tangent space to these two factors are just S_b and S^b, respectively.

As noted earlier, the (interiors of the) faces F_b which correspond to 1-dimensional collision planes S_b already appear as boundary hypersurfaces in the simpler compactification $\overline{\mathfrak{a}}$.

Even if M itself is an irreducible symmetric space, the symmetric spaces $F_s(\gamma)$ which appear in these subsystems may well be reducible. On the algebraic level, this occurs if there is an orthogonal decomposition $S^b = \oplus(S^b)_j$ so that each element of Λ_b lie in one of the summands. An orthogonal partition of roots is the same as an orthogonal partition of simple roots (see [10, Section 10.4]), and this corresponds to the Dynkin diagram decomposing as a disjoint union. This phenomenon occurs already in our standard examples $\mathrm{SL}(n+1)/\mathrm{SO}(n+1)$. In fact, to every possible partition $m_1 + \ldots + m_k = \ell \leq n$ one associates the subsystem

$$\mathbb{R}^{n-\ell} \times \prod_{j=1}^{k} \mathrm{SL}(m_j+1)/\mathrm{SO}(m_j+1).$$

Thus, for example, the subsystems of $\mathrm{SL}(3)/\mathrm{SO}(3)$ are $\mathbb{R}\times\mathbb{H}^2 = \mathbb{R}\times\mathrm{SL}(2)/\mathrm{SO}(2)$, while the two different rank 2 models $\mathbb{R}\times\mathrm{SL}(3)/\mathrm{SO}(3)$ and $\mathbb{R}\times\mathbb{H}^2\times\mathbb{H}^2$, and also the rank 1 model $\mathbb{R}^2\times\mathbb{H}^2$, comprise the subsystems of $\mathrm{SL}(4)/\mathrm{SO}(4)$.

2.8.2. *Analytic subsystems.* We now discuss the subsystem Hamiltonians, and the behaviour of Δ_{rad} near the faces of $\widetilde{\mathfrak{a}}$. Set

$$\rho_b = \frac{1}{2}\sum_{\alpha\in\Lambda_b^+} m_\alpha\,\alpha \quad \left(\text{hence} \quad H_{\rho_b}\in S^b\right). \tag{2.13}$$

The lifts of the roots $\alpha\in\Lambda^+\setminus\Lambda_b^+$ to $\widetilde{\mathfrak{a}}$ tend to $+\infty$ everywhere on the closed face F_b, so that the corresponding terms $(\coth\alpha - 1)H_\alpha$ in Δ_{rad} decay rapidly there and thus are negligible on that face. More precisely, we have the following result.

Lemma 2.3. *Let Z_α be the closure of $\alpha^{-1}((-\infty,0])$ in $\hat{\mathfrak{a}}$. Then*

$$\coth\alpha - 1 \in \mathcal{C}^\infty(\mathfrak{a}\setminus\alpha^{-1}(-\infty,0])$$

extends to an element of $\mathcal{C}^\infty(\hat{\mathfrak{a}}\setminus Z_\alpha)$ that vanishes to infinite order at $\partial\hat{\mathfrak{a}}\setminus\partial Z_\alpha$. Thus, if $\chi\in\mathcal{C}^\infty(\hat{\mathfrak{a}})$ with $\operatorname{supp}\chi\cap Z_\alpha=\emptyset$, then $\chi(\coth\alpha-1)\in\dot{\mathcal{C}}^\infty(\hat{\mathfrak{a}})$, i.e. it vanishes to infinite order at $\partial\hat{\mathfrak{a}}$.

Proof. The function $x\mapsto\alpha(x)/|x|$, $x\in\mathfrak{a}\setminus\{0\}$, is homogeneous degree zero, so it extends to a smooth function on $\hat{\mathfrak{a}}\setminus\{0\}$, and its restriction to $\partial\hat{\mathfrak{a}}\setminus\partial Z_\alpha$ is strictly positive. It is immediate that $e^{-\alpha(x)} = \exp\left(-\frac{\alpha(x)}{|x|}\,|x|\right)$ is smooth and rapidly decreasing in $\hat{\mathfrak{a}}\setminus Z_\alpha$, hence the statements for $\coth\alpha - 1 = \frac{2e^{-2\alpha}}{1-e^{-2\alpha}}$ also follow. □

Note that if $\alpha\in\Lambda^+\setminus\Lambda_b^+$, then in particular $C_{b,\mathrm{reg}}\subset\hat{\mathfrak{a}}\setminus Z_\alpha$, so $\coth\alpha-1$ is Schwartz in a neighbourhood of $C_{b,\mathrm{reg}}$ in $\hat{\mathfrak{a}}$. In other words, there is a conic neighbourhood of $S_{b,\mathrm{reg}}$ in $\mathfrak{a}$ on which $\coth\alpha-1$ is Schwartz.

We now return to Δ_{rad}. After subtracting

$$E_b = \sum_{\alpha\in\Lambda^+\setminus\Lambda_b^+}(\coth\alpha-1)H_\alpha.$$

the remaining terms

$$L_b = \Delta_{S_b} + 2(H_\rho - H_{\rho_b}) + \Delta_{S^b} + 2H_{\rho_b} + \sum_{\alpha\in\Lambda_b^+} m_\alpha(\coth\alpha-1)H_\alpha. \tag{2.14}$$

Proposition 2.4. *For each $b \in I \setminus \{*\}$ there is a decomposition*

$$L_b = T_b + \Delta_{b,\mathrm{rad}},$$

where the first term is a constant coefficient elliptic operator on S_b and the second is the radial Laplacian for the noncompact symmetric space Σ^b, which has rank strictly less than n.

Proof. The first summand, T_b, is the sum of the first two terms in (2.14), and $\Delta_{b,\mathrm{rad}}$ is the sum of the remaining three. Since Λ_b is a root system on S^b, it is clear that

$$\Delta_{\mathrm{rad},b} := \Delta_{S^b} + 2H_{\rho_b} + \sum_{\alpha\in\Lambda_b^+} m_\alpha(\coth\alpha - 1)H_\alpha \tag{2.15}$$

is indeed the radial part of the Laplacian on a symmetric space of lower rank. Thus it remains only to prove that the vector appearing as the first order term in T_b,

$$H_\rho - H_{\rho_b} = \frac{1}{2}\sum_{\alpha\in\Lambda^+\setminus\Lambda_b^+} m_\alpha H_\alpha, \tag{2.16}$$

is an element of S_b, as claimed. To prove this, note first that if β is a simple root, with corresponding Weyl group element w_β (the reflection across W_β) and α is a positive root which is linearly independent from β, then $w_\beta^*(\alpha)$ is again a positive root; for, α is nonnegative and not identically vanishing on $W_\beta \cap \overline{C^+}$, and w_β fixes W_β pointwise, hence $w_\beta^*\alpha$ is also nonnegative and not identically vanishing on this same set, hence must be positive on C^+, which is a characterization of positive roots. Next, clearly $H_{w_\beta^*\alpha} = w_\beta(H_\alpha)$ and so

$$H_\alpha + H_{w_\beta^*(\alpha)} \in W_\beta.$$

In addition, $m_{w_\beta{}^*\alpha} = m_\alpha$. Now let $\{\alpha_j : j \in J_b\}$ be an enumeration of the simple roots in Λ_b^+, and write $w_j = w_{\alpha_j}$. Then w_j^* preserves the subsets Λ_b, hence also $\Lambda \setminus \Lambda_b$ and $\Lambda^+ \setminus \Lambda_b^+$ because α_j is linearly independent from any of the elements in these last two sets. Therefore (2.16) is a sum over w_j orbits, where each orbit consists of one or two elements: if it consists of just one element α, then $H_\alpha \in W_{\alpha_j}$, and if it consists of two elements α and $\alpha' = w_j^*\alpha$, then $m_\alpha H_\alpha + m_{\alpha'}H_{\alpha'}$ also lies in H_{α_j}. Hence (2.16) also lies in W_{α_j}. This is true for every $j \in J_b$, and the claim follows. □

In summary, we have made precise that Δ_{rad} is locally – in a neighbourhood of the lift of $C_{b,\mathrm{reg}}$ to $\tilde{\mathfrak{a}}$ – the sum of a product model, L_b, and an error term E_b.

We remark that such a neighbourhood is diffeomorphic to an open subset in the tilde-compactification of $\mathfrak{a}$ with collision planes given by $S_b \times (S_c \cap S^b)$ and $\{0\}$ as S_c runs over all collision planes satisfying $S_c \supset S_b$. In particular, if one studies the asymptotics of the Green function, one can paste the asymptotics of the local model operator Green functions directly from the model space to $\tilde{\mathfrak{a}}$.

3. Invariant smooth functions and localization on the compactified spaces

3.1. **Invariant smooth functions.** As already discussed in §2.1, every $g \in G$ decomposes into a product $g = k_1 a k_2$, where $k_1, k_2 \in K$ and $a \in \mathcal{A}$; the middle factor is determined up to translation by an element of W, and in particular is unique if we require it to lie in $\overline{\mathcal{A}^+}$. This defines a map $\pi : M \to \overline{\mathcal{A}^+}$. If h is any (e.g. measurable) function on $\overline{\mathfrak{a}^+}$, or equivalently, a W-invariant function on $\mathfrak{a}$, then its pullback $\pi^* h$ is a K-invariant function on $G/K = M$. (As usual, we are identifying $\mathcal{A}$ with $\mathfrak{a}$.) Conversely, K-invariant functions on M restrict to W-invariant functions on $\mathfrak{a}$, and therefore π^* induces an equivalence between these spaces.

It will be important for us to know whether π^* yields an equivalence between functions with higher regularity. Thus, for example, it is clear that π^* induces an isomorphism between continuous W- and K- invariant functions, and also between L^2_{loc} invariant functions, though here we must use the degenerate measure on $\mathfrak{a}$ induced by pushforward by π_* of a smooth invariant smooth measure on M. Somewhat more generally, π is a Riemannian submersion since the K-orbits are orthogonal to $\mathcal{A}$ and the metric is invariant on both fibre and base. Hence it is distance-decreasing, i.e. $d(\pi(x), \pi(y)) \leq d(x, y)$ for any $x, y \in M$; therefore π is Lipschitz, and π^* gives an isomorphism between invariant functions which are locally Lipschitz. The following result, however, is less obvious.

Proposition 3.1. *The map $\pi^* : \mathcal{C}^\infty(\mathfrak{a})^W \to \mathcal{C}^\infty(M)^K$ is an isomorphism.*

Proof. The easy direction is that the restriction of any $f \in \mathcal{C}^\infty(M)^K$ to $\mathcal{A}$ is in $\mathcal{C}^\infty(\mathfrak{a})^W$. In fact, the inclusion map $\iota : \mathcal{A} \hookrightarrow M$ is smooth, so if $f \in \mathcal{C}^\infty(M)$ then $\iota^*(f) \in \mathcal{C}^\infty(\mathfrak{a})$. Moreover, since W is the quotient of the normalizer in K of $\mathcal{A}$ by its centralizer, ι commutes with the action of W, and so $\iota^* : \mathcal{C}^\infty(M)^K \to \mathcal{C}^\infty(A)^W$.

To prove the converse, we use induction on the rank n. Suppose the result has been proved for all symmetric spaces of rank strictly less than n. Fix $p \in \overline{\mathcal{C}^+} \setminus \{0\}$, so $p \in S_{b,\mathrm{reg}}$ for some $b \in I \setminus \{*\}$. As explained in §2.6, there is a neighbourhood $\mathcal{U}$ of p in $\mathfrak{a}$ such that the preimage $\pi^{-1}(\mathcal{U})$ in M is a bundle over K/K^b with fiber an open neighbourhood of (o, p) in $\Sigma^b \times S_b$. The subgroup $W^b \subset W$ generated by roots $\alpha \in \Lambda_b$ is naturally identified with the Weyl group of Σ^b. Now suppose that $u \in \mathcal{C}^\infty(\mathfrak{a})^W$. Then the restriction of u to $\mathcal{U}$ can be considered as a smooth W^b-invariant function on (some neighbourhood of a point $(0, p)$ in) $S^b \oplus S_b$. By the inductive hypothesis, $\pi^* u$ can be identified with a smooth K^b-invariant function on a neighbourhood of $(o, p) \in \Sigma^b \oplus S_b$. Since b is arbitary, this proves that $\pi^* u \in \mathcal{C}^\infty(M \setminus \{o\})^K$.

It remains to prove that $\pi^* u$ is also smooth near o. At the same time we must also start the induction, proving that $\pi^* u$ is smooth on M for symmetric spaces of rank one, but since the only issue in that case is to prove smoothness at o, this is the same argument.

We proceed as follows. Let L be the operator on $\mathfrak{a}$ induced by Δ_{rad} on $\mathfrak{a}_{\mathrm{reg}}$; according to Lemma 2.1, L preserves $\mathcal{C}^\infty(\mathfrak{a})^W$. We have already remarked that since $u \in \mathcal{C}^\infty(\mathfrak{a})^W$ is locally Lipschitz, the same is true of $\pi^* u$. Moreover, $Lu \in \mathcal{C}^\infty(\mathfrak{a})^W$, so $\pi^*(Lu)$ is also locally Lipschitz on M. By the induction, $\pi^*(Lu)$ agrees with the smooth function $f = \Delta(\pi^* u)$ away from o. Hence $\Delta(\pi^* u)$ is a distribution differing from the locally Lipschitz function $\pi^*(Lu)$ by a distribution supported at o. However, $\nabla \pi^* u \in L^\infty_{\mathrm{loc}}$, so in particular $\pi^* u \in H^1_{\mathrm{loc}}$, which implies that $\Delta(\pi^* u) \in H^{-1}_{\mathrm{loc}}$. Furthermore, since it is locally Lipschitz, $\pi^*(Lu) \in H^1_{\mathrm{loc}}$ too. Therefore the difference $g = \Delta(\pi^* u) - \pi^*(Lu) \in H^{-1}_{\mathrm{loc}}$. If $\dim M \geq 2$, no element of H^{-1}_{loc} can be supported at o, so $g = 0$. If $\dim M = 1$, then the K is finite and the same conclusion is trivial.

We have now proved that $\Delta\pi^* u$ is locally Lipschitz, and $\Delta\pi^* u = \pi^*(Lu)$. Now repeat the argument with u replaced by Lu to conclude that $\Delta^j \pi^* u$ is locally Lipschitz for every $j \geq 1$. By elliptic regularity, $\pi^* u \in \mathcal{C}^\infty(M)$, and this completes the proof. □

This result extends to the compactifications, as is easily seen from the proof of Proposition 3.1: in the inductive step, we merely need to compactify the base space S_b of the family.

Proposition 3.2. *The map* π^* *gives isomorphisms* $\mathcal{C}^\infty(\overline{\mathfrak{a}})^W \to \mathcal{C}^\infty(\overline{M})^K$ *and* $\mathcal{C}^\infty(\tilde{\mathfrak{a}})^W \to \mathcal{C}^\infty(\tilde{M})^K$.

3.2. **Invariant partitions of unity.** We now introduce W-invariant partitions of unity on $\mathfrak{a}$ which are compatible with the structures of the compactifications $\hat{\mathfrak{a}}$ and $\overline{\mathfrak{a}}$. The lifts of these partitions of unity are of course K-invariant partitions of unity on M compatible with the structures of the corresponding compactifications.

Each (open) face S_b^+ of the closed positive Weyl chamber $\overline{\mathcal{C}^+}$ in $\mathfrak{a}$ is an open set in a unique S_b, $b \in I$, and therefore we may index the set of all such faces S_b^+ by a subset $I^+ \subset I$.

We first consider invariant partitions of unity on $\mathfrak{a}$:

Definition 3.3. A partition of unity $\{\chi_b : \ b \in I^+\}$ on $\overline{\mathcal{C}^+}$ is W-adapted if each χ_b is the restriction to $\overline{\mathcal{C}^+}$ of some $\chi_b' \in \mathcal{C}^\infty(\mathfrak{a})^W$, and moreover if $\operatorname{supp}\chi_b \cap S_c^+ = \emptyset$ except when $S_b^+ \subset S_c$.

Remark 3.4. Since $\sum \pi^*\chi_b' = \pi^*(\sum \chi_b') = 1$, the lifts $\pi^*\chi_b'$ are a smooth K-invariant partition of unity on M.

No conditions have been imposed on the χ_b at infinity, so this partition of unity is only useful for studying local properties. To go further, let $\widehat{\mathcal{C}^+}$ be the closure of $\overline{\mathcal{C}^+}$ in the radial compactification $\hat{\mathfrak{a}}$.

Definition 3.5. A partition of unity $\{\chi_b : \ b \in I^+\}$ on $\widehat{\mathcal{C}^+}$ is $(W, \hat{\mathfrak{a}})$-adapted if

(i) each χ_b is the restriction to $\widehat{\mathcal{C}^+}$ of an element of $\mathcal{C}^\infty(\hat{\mathfrak{a}})^W$,
(ii) $\operatorname{supp}\chi_*$ is a compact subset of $\mathfrak{a}$, and
(iii) $\operatorname{supp}\chi_b \cap \widehat{S_c^+} = \emptyset$ unless $S_b^+ \subset S_c$; here $\widehat{S_c^+}$ is the closure of S_c^+ in $\hat{\mathfrak{a}}$.

The restriction that χ_b be supported sufficiently near to S_b^+, i.e. (iii), ensures that $L_{b,\mathrm{rad}}$ is a good model for Δ_{rad} on its support. On the other hand, (ii) guarantees that the partition of unity is not trivial: i.e. that $\chi_* \not\equiv 1$.

Lemma 3.6. *There exists a* $(W, \hat{\mathfrak{a}})$*-adapted partition of unity.*

Proof. We first construct a partition of unity on $\hat{\mathfrak{a}}$ with the appropriate support properties, then average it over W.

For any root α, let $\widehat{W_\alpha}$ denote the closure of the wall W_α in $\hat{\mathfrak{a}}$. Also, set

$$\widehat{W_{\alpha,\pm}} = \overline{\alpha^{-1}(\mathbb{R}^\pm)} \setminus \widehat{W_\alpha};$$

this is the closure in $\hat{\mathfrak{a}}$ of the set where $\alpha > 0$, respectively $\alpha < 0$, minus the closure of the wall. We say that $\alpha > 0$ on $\widehat{W_{\alpha,+}}$ and $\alpha < 0$ on $\widehat{W_{\alpha,-}}$ and $\alpha = 0$ on $\widehat{W_\alpha}$.

Each face of each Weyl chamber is defined by a map $\mu : \Lambda \to \{0, +, -\}$, corresponding to whether each root is > 0, < 0 or $= 0$ on that face. Denote the space of all such maps by $\mathcal{P}$. Certain $\mu \in \mathcal{P}$ correspond to empty faces (for instance if one requires that both α and $-\alpha$ are positive), so we let $\mathcal{P}_0$ be the subset of μ for which the corresponding face is nonempty. To any $\mu \in \mathcal{P}_0$ such that $\mu(\alpha) \neq 0$ for at least one α we associate the relatively open set

$$U_\mu = \left(\bigcap\{\widehat{W_{\alpha,+}} : \ \mu(\alpha) > 0\}\right) \cap \left(\bigcap\{\widehat{W_{\alpha,-}} : \ \mu(\alpha) < 0\}\right) \subset \hat{\mathfrak{a}};$$

with $*$ corresponding to the map $\mu \equiv 0$ we also set $U_* = \mathfrak{a}$.

The collection $\mathcal{U} = \{U_\mu\}$ is an open cover of $\hat{\mathfrak{a}}$, and we choose a partition of unity $\{\psi_\mu\}$ subordinate to it. Every $w \in W$ is an endomorphism of $\mathfrak{a}$, and extends to a diffeomorphism of $\hat{\mathfrak{a}}$. To each such w, if $\alpha \in \Lambda$, then $w_*\mu$ is the map which assigns to $w^*\alpha$ the value $\mu(\alpha)$. Finally, let

$$\phi_\mu = \frac{1}{|W|} \sum_{w \in W} w^* \psi_\mu.$$

Then $\sum_\mu \phi_\mu = 1$ and each ϕ_μ is clearly W-invariant.

If the face corresponding to some μ is not contained in $\overline{C^+}$, then $U_\mu \cap \widehat{C^+} = \emptyset$. Indeed, for any such μ there is a positive root α such that $\mu(\alpha) < 0$, so $\alpha < 0$ on U_μ, which means that U_μ does not intersect the closed positive chamber.

Note also that for any $\mu \in \mathcal{P}_0$, there is a unique $\mu_+ = w_*\mu$ which is ≥ 0 on all positive roots. Since $w^*\psi_\mu$ is supported in $w^{-1}(U_\mu) = U_{w_*\mu}$, we have $\operatorname{supp} w^*\psi_\mu \cap \widehat{\mathcal{C}^+} = \emptyset$ unless $w_*\mu = \mu_+$.

Now suppose that S_b^+ is a face of $\overline{C^+}$. Clearly $S_b \subset S_c$ if and only if for every root α, $\alpha \equiv 0$ on S_c implies $\alpha \equiv 0$ on S_b. Thus if $S_b \not\subset S_c$, then there is a root α, which we may assume is positive, which vanishes identically on S_c but not on S_b. In particular, if ν is the map corresponding to $b \in I^+$, then $\nu(\alpha)$ is positive (since $b \in I^+$), hence non-zero, and so $U_\nu \cap \widehat{S_c^+} = \emptyset$ by the definition of U_ν.

Finally, combine each W-orbit of ϕ_μ into a single term

$$\chi_b = \chi_\nu = \sum_{w \in W} \phi_{w_*\nu}.$$

Now, for $w \in W$, $\operatorname{supp} w^*\psi_{v_*\nu} \cap \widehat{\mathcal{C}^+} = \emptyset$ unless $w^*v^*\nu = (v^*\nu)_+ = \nu_+ = \nu$ since ν is ≥ 0 on positive roots. On the other hand, if $w^*v^*\nu = \nu$, and c is as in the previous paragraph, then $\operatorname{supp} w^*\psi_{v^*\nu} \cap \widehat{S_c^+} \subset U_\nu \cap \widehat{S_c^+} = \emptyset$. Therefore, for every $v, w \in W$, $\operatorname{supp} w^*\psi_{v^*\nu} \cap \widehat{\mathcal{C}^+} = \emptyset$. This shows that $\operatorname{supp} \chi_b \cap \widehat{S_c^+} = \emptyset$, which finishes the proof. □

Definition 3.7. A partition of unity $\{\chi_b : \ b \in I^+\}$ on $\overline{\mathcal{C}^+}$ is $(W, \overline{\mathfrak{a}})$-adapted if

(i) each χ_b is the restriction to $\overline{\mathcal{C}^+}$ of an element in $\mathcal{C}^\infty(\overline{\mathfrak{a}})^W$,
(ii) $\operatorname{supp} \chi_b \cap \overline{S_c^+} = \emptyset$ unless $S_b^+ \subset S_c$ (where $\overline{S_c^+}$ is the closure of S_c^+ in $\overline{\mathfrak{a}}$), and
(iii) $\operatorname{supp} \chi_b \subset S_b \cap \Omega_b$, where Ω_b is a compact subset of S^b (and in particular, χ_* has compact support since $S_* = \{0\}$).

Lemma 3.8. *There exists a $(W, \overline{\mathfrak{a}})$-adapted partition of unity.*

The proof proceeds just as for the $(W, \hat{\mathfrak{a}})$-adapted case, and so we omit it.

Remark 3.9. Note that if $\{\chi_b : \ b \in I^+\}$ is a $(W, \overline{\mathfrak{a}})$-adapted partition of unity, then there exists $T = (T_1, \ldots, T_n)$ with all $T_j > 0$ such that $\operatorname{supp} \chi_0 \subset \mathcal{O}(T)$, so χ_0 localizes away from the walls. In addition, all the N-body features of the analysis are already present on $\operatorname{supp} \chi_0$.

4. Differential operators, function spaces and mapping properties

In this section we explain the appropriate spaces of differential operators and functions of finite regularity that are used later.

We start with differential operators, or more specifically, K-invariant operators acting on K-invariant function spaces. If P is such an operator and P_{rad} its radial part, then since $\mathcal{C}_c^\infty(M)^K$ is identified with $\mathcal{C}_c^\infty(\mathfrak{a})^W$, and $\mathcal{C}_c^\infty(M)^K$ is dense in every function space we wish to study, we can regard P_{rad} either as a map $\mathcal{C}_c^\infty(M)^K \to \mathcal{C}_c^\infty(M)^K$ (i.e. as the restriction of P), or as a map $\mathcal{C}_c^\infty(\mathfrak{a})^W \to \mathcal{C}_c^\infty(\mathfrak{a})^W$. In the former case, P_{rad} is a differential operator on M with $\mathcal{C}^\infty$ coefficients, while in the latter case, P_{rad} is a differential operator whose coefficients on $\mathfrak{a}_{\mathrm{reg}}$ are smooth, hence gives a map $\mathcal{C}^\infty(\mathfrak{a}_{\mathrm{reg}})^W \to \mathcal{C}^\infty(\mathfrak{a}_{\mathrm{reg}})^W$, which restricts to a map $\mathcal{C}_c^\infty(\mathfrak{a})^W \to \mathcal{C}_c^\infty(\mathfrak{a})^W$. One could define the appropriate space of differential operators directly on $\mathfrak{a}$, but one must take care to see their uniformity near the walls. We proceed instead by identifying functions on neighbourhoods of the walls in $\mathfrak{a}$ with neighbourhoods in a product model.

Let $\{\chi_b : b \in I^+\}$ be a $(W, \overline{\mathfrak{a}})$-adapted partition of unity, and fix diffeomorphisms

$$\Psi_b : \operatorname{supp} \chi_b \hookrightarrow S_b \oplus S^b.$$

Then to any W^b-invariant function u on $S_b \oplus S^b$, we can associate a K^b-invariant function $\tilde{u}_b$ on $S_b \times \Sigma^b$, and conversely the restriction of such a K^b invariant function to $S_b \oplus S^b$ is W^b-invariant. If $\operatorname{supp} u \subset S_b \times \mathcal{V}_b$, then $\operatorname{supp}(\tilde{u}_b) \subset S_b \times \widetilde{\mathcal{V}}_b$, where $\widetilde{\mathcal{V}}_b$ is a bounded set containing the origin in Σ^b.

The operators we shall single out are generated by translation invariant operators on S_b and arbitrary differential operators on the bounded set $\widetilde{\mathcal{V}}_b \subset \Sigma^b$. We also require the operators of multiplication by functions in both $\mathcal{C}^\infty(\hat{\mathfrak{a}})^W$ and $\mathcal{C}^\infty(\overline{\mathfrak{a}})^W$, since elements of the former are required in the partition of unity patching the local models, while the form of the Laplacian requires the latter; these requirements suggest that we allow multiplication by functions in $\mathcal{C}^\infty(\widetilde{M})^K \equiv \mathcal{C}^\infty(\widetilde{\mathfrak{a}})^W$.

Definition 4.1. The space $\mathrm{Diff}^m_{\mathrm{ss},o}(M)$ consists of all differential operators $P : \mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(M)$ of order m which are K-invariant, and such that for each $b \in I^+$, the K-radial part P_{rad} of P, restricted to functions supported in $\pi^{-1}(\operatorname{supp}\chi_b)$, is the K^b-radial part Q_{rad} of a differential operator Q on $S_b \times \widetilde{\mathcal{V}}_b$ which is a linear combination of products of translation invariant operators on S_b and differential operators on $\widetilde{\mathcal{V}}_b$, with coefficients in $\mathcal{C}^\infty(\widetilde{S_b} \times \widetilde{\mathcal{V}}_b)$.

Remark 4.2. The subscript o has been included in this notation because this space of operators depends on the choice of origin in M. We note also that this definition only restricts the behaviour of these operators near infinity. Finally, recall from Remark 3.9 that χ_0 (considered as a function on $\overline{\mathcal{C}^+}$) is supported in $\mathcal{O}(T)$ for some T with all $T_j > 0$. Thus, for $b = 0$ the requirement is that P_{rad} restricted to $\mathcal{O}(T)$ is a linear combination of translation invariant differential operators in $\mathfrak{a}$ with coefficients in $\mathcal{C}^\infty(\widetilde{\mathfrak{a}})$. Apart from the localization to $\mathcal{O}(T)$, this is *exactly* the definition of N-body differential operators $\mathrm{Diff}^*_{\mathrm{sc}}(\hat{\mathfrak{a}}, \mathcal{C})$, $\mathcal{C} = \mathcal{S} \cap \partial\hat{\mathfrak{a}}$, in [24].

The use of the product spaces $S_b \times \widetilde{\mathcal{V}}_b$ is motivated by the results of § 2.6; see in particular Remark 2.2.

We now discuss the associated L^2-based Sobolev spaces. The basic L^2 space is, of course, $L^2(M, dg)^K$, which is identified with an L^2-space with respect to the degenerate measure on $\mathfrak{a}$, $dg_0 = \pi_* dg := \eta\, da$ where $\pi : M \to \mathcal{C}^+$; note that η extends to be W-invariant function on $\mathfrak{a}$. There is an explicit formula [8, Ch. 1, Theorem 5.8]

$$\eta(a) = \prod_{\alpha\in\Lambda^+} (\sinh \alpha(a))^{m_\alpha}, \qquad a \in C^+. \tag{4.1}$$

Notice that $\eta(a)$ is $\mathcal{C}^\infty$ and strictly positive on $\mathfrak{a}_{\mathrm{reg}}$, but degenerates like various powers of the distance function along the Weyl chamber walls, i.e. where various roots α vanish. Then

$$L^2(M, dg)^K \equiv L^2(\mathcal{C}^+, dg_0) \equiv L^2(\mathfrak{a}, \frac{1}{|W|}\, dg_0)^W$$

as Hilbert spaces; of course, the norms of the last terms are equivalent without the constant factor $|W|^{-1}$.

As M is a non-compact space, there are various spaces of K-invariant Sobolev functions that we can associate to it. We need the spaces that correspond to $\mathrm{Diff}_{\mathrm{ss},o}(M)$, which was in turn constructed to accommodate both the Laplacian and multiplication by cutoffs in $\mathcal{C}^\infty(\hat{\mathfrak{a}})$. For $b \in I^+$, we let

$$\eta_b(a) = \prod_{\alpha\in\Lambda_b^+} (\sinh \alpha(a))^{m_\alpha}, \qquad a \in C^+,$$

note that on $\mathrm{supp}\,\chi_b$ we can identify $\eta_b\, da^b$ with the push-forward of the Riemannian measure dg_b on Σ^b to the positive chamber of S^b. Moreover, the other positive roots $\alpha \in \Lambda^+ \setminus \Lambda_b^+$ tend to $+\infty$ on $\mathrm{supp}\,\chi_b$, so $e^{-2(\rho-\rho_b)} \prod_{\alpha\in\Lambda^+\setminus\Lambda_b^+} (\sinh \alpha(a))^{m_\alpha}$ is bounded from below and above by positive constants. Correspondingly, for functions in $L^2(M, dg)^K$ supported in $\mathrm{supp}\,\chi_b$, the $L^2(M, dg)$-norm is equivalent to the $L^2(S_b \times \Sigma^b; e^{2(\rho-\rho_b)}\, da_b\, dg_b)$-norm; here da_b is the Euclidean density on S_b. We now define the Sobolev spaces as follows.

Definition 4.3. The space $H^s_{\mathrm{ss},o}(M)^K$ is the set of distributions $u \in \mathcal{D}'(M)^K \equiv \mathcal{D}'(\mathfrak{a})^W$ with the property that $e^{\rho-\rho_b}\, \widetilde{((\Psi_b)_*(\chi_b u))}_b \in H^s(S_b \times \Sigma^b)$. (Because the support is bounded in the second factor, there are no subtleties involving noncompact supports in this condition.)

Remark 4.4. Continuing Remark 4.2, note that for $b = 0$ the requirement is simply that $e^\rho \chi_0 u \in H^s(\mathfrak{a})$, i.e. $\chi_0 u$ is in the weighted Sobolev space $e^{-\rho} H^s(\mathfrak{a})$ (where $H^s(\mathfrak{a})$ is the standard Sobolev space on the vector space $\mathfrak{a}$).

Remark 4.5. We could have equally well defined these adapted classes of differential operators and Sobolev spaces using the identification of neighbourhoods of the supports of elements of a $(\hat{\mathfrak{a}}, W)$-adapted partition of unity, i.e. by working on conic neighbourhoods of the S_b. This would require that definitions be made inductively on the rank, since we would no longer be working in compact subsets of the subsystems Σ^b.

If $s \geq 0$ is an integer, this means that for any $A \in \mathrm{Diff}^k_{\mathrm{ss},o}(M)$ with $k \leq s$,

$$Au \in L^2(M, dg)^K.$$

Indeed, by the definition of $\mathrm{Diff}_{\mathrm{ss},o}(M)$, the latter statement is equivalent to requiring that for any translation invariant differential operator P of order $k \geq 0$ on

S_b and for any differential operator Q of order $l \geq 0$ on Σ^b, with $k + l \leq s$,

$$PQ\left((\Psi_b)_*(\chi_b u)\right)\tilde{}_b \in L^2(e^{2(\rho-\rho_b)}\, da_b\, dg_b).$$

Since commuting the weight through P introduces lower order differential operators, this is easily seen to be equivalent to

$$PQe^{\rho-\rho_b}\left((\Psi_b)_*(\chi_b u)\right)\tilde{}_b \in L^2(da_b\, dg_b),$$

for all P and Q as above, which is the definition of the Sobolev spaces.

A key property that a parametrix G for $\Delta_{\text{rad}} - \lambda$ should have is that its error $F = (\Delta_{\text{rad}} - \lambda)G - \text{Id}$ should be a compact operator, say on $L^2(M, dg)^K$. We can achieve this by showing that F maps into a positive order Sobolev space with additional decay at infinity. Thus, we also consider spaces of functions on $\widetilde{\mathfrak{a}}$ with some specified rate of decay at the boundary. To this end, we introduce the total boundary defining function

$$x = \prod_{b \in I \setminus \{*\}} x_b,$$

where x_b is a defining function for the face $\widetilde{F_b}$ of $\widetilde{\mathfrak{a}}$. Note that $\hat{x}$ agrees with x up to a smooth non-vanishing positive factor, as follows by considering $\widetilde{\mathfrak{a}}$ as a blow-up of $\hat{\mathfrak{a}}$.

Supposing that x is W-invariant, we then define

$$x^\delta H^s_{\text{ss},o}(M)^K = \{u = x^\delta v : v \in H^s_{\text{ss},o}(M)^K\}$$

(which by the remark above is the same as $\hat{x}^\delta H^s_W(\widetilde{\mathfrak{a}})$).

Proposition 4.6. *For any $s, \delta \in \mathbb{R}$, $\text{Diff}^m_{\text{ss},o}(M) : x^\delta H^s_{\text{ss},o}(M)^K \to x^\delta H^{s-m}_{\text{ss},o}(M)^K$.*

Proof. Both the Sobolev spaces and the differential operators are defined by localization to $S_b \times \widetilde{\mathcal{V}}_b$, and on these the claims are clear. □

It is crucial for us that parametrix constructions can be localized on $\hat{\mathfrak{a}}$. This is reflected by the following proposition.

Proposition 4.7. *The multiplication operators $\phi \in \mathcal{C}^\infty(\hat{\mathfrak{a}})^W$ commute with operators $P \in \text{Diff}^k_{\text{ss},o}(M)$ to top order, i.e. $[P, \phi] \in x\,\text{Diff}^{k-1}_{\text{ss},o}(M)$. Thus, $[P, \phi] : x^\delta H^{s+m-1}_{\text{ss},o}(M)^K \to x^{\delta+1} H^s_{\text{ss},o}(M)^K$.*

Remark 4.8. The analogue of this result has been widely used in N-body scattering. There is a much larger class of (pseudo-)differential operators which commute to top order with every $P \in \text{Diff}^k_{\text{ss},o}(M)$, and which can be used to microlocalize, see [24].

Proof. Using a partition of unity, we assume that P is supported in $\pi^{-1}(\text{supp}\,\chi_b)$. Valid local coordinates on $\hat{\mathfrak{a}}$ near $\hat{S}_b$ are of the form $\frac{\alpha_j(a)}{|a|}$, $a \in \mathfrak{a}$, where the α_j are linearly independent simple roots that vanish on S_b, as well as coordinates on $\hat{S}_b$. Thus, in a neighbourhood of $\hat{S}_b$ (which includes $\text{supp}\,P$)

$$\phi = \phi|_{\hat{S}_b} + \sum_j \frac{\alpha_j(a)}{|a|}\phi_b,$$

with ϕ_b smooth in this open subset of $\hat{\mathfrak{a}}$. In particular, its commutator with P is in $\text{Diff}^{k-1}_{\text{ss},o}(M)$. Using this expansion now it is straightforward to complete the proof. □

Specializing these results to the Laplacian, we deduce that for any $s, \delta \in \mathbb{R}$ and $\lambda \in \mathbb{C}$,

$$\Delta_{\mathrm{rad}} - \lambda : x^\delta H^{s+2}_{\mathrm{ss},o}(M)^K \longrightarrow x^\delta H^s_{\mathrm{ss},o}(M)^K.$$

Ultimately, of course, we are interested in inverting this operator, and as usual, this will rely on its ellipticity.

Definition 4.9. We say that $P \in \mathrm{Diff}_{\mathrm{ss},o}(M)$ is *radially elliptic* if for every $b \in I^+$, there is an operator $Q = Q_b \in \mathrm{Diff}^m_{\mathrm{ss},o}(S_b \times \Sigma^b)$ as in Definition 4.1 that is symbol-elliptic.

Remark 4.10. We emphasize that symbol-ellipticity in $\mathrm{Diff}^m_{\mathrm{ss},o}(S_b \times \Sigma^b)$ is a *uniform* condition near infinity in S_b.

In particular, for $b = 0$, such a differential operator has the form $\sum_{|\gamma| \leq m} p_\gamma(a) D^\gamma$, with p_γ smooth on the closure of $\mathcal{O}(T)$ in $\widetilde{\mathfrak{a}}$, $T_j > 0$ for all j. Symbol ellipticity then is the requirement that $\sum_{|\gamma|=m} p_\gamma(a)\xi^\gamma$ never vanish for (a, ξ) in the closure of $\mathcal{O}(T) \times \mathfrak{a}^*$ in $\widetilde{\mathfrak{a}} \times \mathfrak{a}^*$.

Clearly, Δ_{rad} is radially elliptic. Indeed, we can take $Q_b = T_b + \Delta_{\Sigma^b}$. Thus, one can use the standard parametrix construction for $\Delta_{\mathrm{rad}} - \lambda$; indeed, even the standard *large spectral parameter* construction works, i.e. we can precisely analyze $|\lambda| \to \infty$.

5. Complex scaling

As explained in the introduction, there are two main tools in our proof of the analytic continuation of Δ_{rad}: construction of the parametrix, which takes place in the b-calculus on $\widetilde{\mathfrak{a}}$, and the method of complex scaling. In this section we focus on the second of these, and shall review this method, which produces a holomorphic family of operators for which the essential spectrum is shifted away from the positive real axis.

The ingredients needed in this procedure are a family of (possibly unbounded) operators U_θ acting on $L^2(\mathfrak{a})^W$, for θ lying in some contractible domain $D \subset \mathbb{C}$, and a dense subspace of 'analytic vectors' $\mathbb{A} \subset L^2(\mathfrak{a})^W$, such that:

(i) $U_0 = \mathrm{Id}$ and for $\theta \in D \cap \mathbb{R}$, U_θ is unitary on $L^2(\mathfrak{a})^W$ and bounded on all Sobolev spaces;
(ii) For $f \in \mathbb{A}$, the map $\theta \to U_\theta f$ extends analytically from $D \cap \mathbb{R}$ to all of D with values in $L^2(\mathfrak{a})^W$;
(iii) For each $\theta \in D$, the subspace $U_\theta \mathbb{A}$ is dense in $L^2(\mathfrak{a})^W$.

By (i), we can define $\Delta_{\mathrm{rad},\theta} = U_\theta \Delta_{\mathrm{rad}} U_\theta^{-1}$ directly when $\theta \in \mathbb{R}$. We shall show below that the coefficients of this operator extend analytically in θ to the sector $|\operatorname{Im}\theta| < \frac{\pi}{2}$; hence for fixed $f \in \mathcal{C}^\infty_c(M)$, $\theta \to \Delta_{\mathrm{rad},\theta} f$ is analytic in this same region. We must actually prove that the family $\Delta_{\mathrm{rad},\theta}$ is analytic of type A, see Proposition 5.4 below. The resolvent of the scaled radial Laplacian, $(\Delta_{\mathrm{rad},\theta} - \lambda)^{-1}$, will be constructed by parametrix methods. From this we can deduce the meromorphic continuation of $R(\lambda)$ from the equality $(\Delta_{\mathrm{rad},\theta} - \lambda)^{-1} = U_\theta R(\lambda) U_\theta^{-1}$, which is initially valid when λ is in the resolvent set common to both operators and θ is real. In fact, we prove only that the matrix element $\langle f, R(\lambda) g\rangle$ continues meromorphically to D whenever $f, g \in \mathbb{A}$; this is sufficient for purposes of spectral theory.

5.1. **Complex dilations.** Let $\mathfrak{p}_{\mathbb{C}}$ denote the complexification of $\mathfrak{p}$ and D some domain in $\mathbb{C}$ containing 0, and define

$$\Phi : D \times \mathfrak{p} \longrightarrow \mathfrak{p}_{\mathbb{C}}; \qquad \Phi(\theta, X) = e^{\theta} X.$$

We also denote that restriction of Φ to $D \times \mathfrak{a} \longrightarrow \mathfrak{a}_{\mathbb{C}}$ by Φ, and often write $\Phi_\theta(X) = \Phi(\theta, X)$. Identifying $\mathfrak{p}$ and M by the exponential map, for $\theta \in \mathbb{R} \cap D$ Φ_θ is the diffeomorphism on M given by dilating by the factor e^θ along geodesic rays emanating from o.

When $\theta \in \mathbb{R}$, the induced family of unitary operators U_θ on

$$L^2(M)^K \equiv L^2(\mathfrak{a}, |W|^{-1}\pi_* dg)^W$$

is defined by

$$(U_\theta f)(a) = (\det\, D_\theta \Phi)^{\frac{1}{2}} f(e^\theta a) = J_\theta^{\frac{1}{2}} (\Phi_\theta^* f)(a),\ a \in \mathfrak{a}; \tag{5.1}$$

the Jacobian prefactor, which is calculated with respect to the density $\pi_* dg = \eta\, da$ in (4.1), makes this map unitary. Explicitly, with $n = \dim \mathfrak{a}$,

$$J_\theta(a) = (\det\, D_\theta \Phi)(a) = w^n \frac{\eta(wa)}{\eta(a)} = w^n \prod_{\alpha \in \Lambda^+} \left(\frac{\sinh(w\alpha(a))}{\sinh(\alpha(a))} \right)^{m_\alpha},\ a \in C^+.$$

Note that J_θ does not vanish for $|\operatorname{Im} \theta| < \frac{\pi}{2}$. The product can be replaced by one over Λ, if m_α is replaced by $m_\alpha/2$, and then the formula is valid on all of $\mathfrak{a}$; this also shows that J_θ is $\mathcal{C}^\infty$ on $\mathfrak{a}$.

While the use of U_θ fits nicely into the Aguilar-Balslev-Combes theory, one could also work with Φ_θ^* directly, which would be closer in spirit to the microlocal complex deformations of Sjöstrand and Zworski [21].

Lemma 5.1. *For $\theta \in \mathbb{R}$, Φ_θ extends to a 'conormal diffeomorphism' of $\overline{\mathfrak{a}}$, in the sense that $\Phi_\theta^* : S^m(\overline{\mathfrak{a}}) \mapsto S^{mw}(\widetilde{\mathfrak{a}})$, where $w = e^\theta$ and $S^m(\overline{\mathfrak{a}})$ denotes the symbol space. In addition, it extends to a diffeomorphism of $\widetilde{\mathfrak{a}}$.*

Proof. The first claim is easy to check since the effect of dilations is that roots α are multiplied by e^θ: $\Phi_\theta^* \alpha(a) = \alpha(e^\theta a) = e^\theta \alpha(a)$, and the negative exponentials of the simple roots define the smooth structure of $\overline{\mathfrak{a}}$ in a neighbourhood of $\overline{C^+}$.

The second claim follows from either description of $\widetilde{\mathfrak{a}}$. Indeed, Φ_θ extends to a diffeomorphism of $\hat{\mathfrak{a}}$, and then lifts to its blow-up $\widetilde{\mathfrak{a}}$. Alternatively, the logarithmic total boundary blow-up replaces the defining functions $e^{-\alpha_j}$ of $\overline{\mathfrak{a}}$ in C^+ by α_j^{-1}, so Φ_θ extends to a diffeomorphism of the this blow-up, which then lifts to $\widetilde{\mathfrak{a}}$. □

Lemma 5.2. *The Jacobian determinant J extends to an analytic nonvanishing function in the region*

$$D = \{\theta \in \mathbb{C} :\ |\operatorname{Im} \theta| < \frac{\pi}{2}\}.$$

In addition, J, $J^{1/2}$ and $J^{-1/2}$ are conormal K-invariant functions on $\overline{M}$, equivalently, conormal W-invariant functions on $\overline{\mathfrak{a}}$.

We shall need a slight generalization of this definition later. Let $\Phi_{\theta,T}$ be a W-invariant diffeomorphism of $\mathfrak{a}$ which is the identity on the ball $B_T(0)$ and equals the dilation by e^θ outside a larger ball, and which depends analytically on θ. For example, fix $T > 0$ and a nondecreasing cutoff function $\phi \in \mathcal{C}^\infty(\mathbb{R}; [0,1])$ which equals 1 near ∞ and vanishes on $[0,T]$, and define

$$\Phi_{\theta,T}(a) = e^{\phi(r)\theta} a;$$

then $\Phi_{\theta,T}(a) = a$ if $|a| \le T$, and $\Phi_{\theta,T}(a) = e^\theta a$ for $|a| \ge T' > T$, and $\theta \mapsto \Phi_{\theta,T}(a)$ is analytic. It is clear that $\Phi_{\theta,T}$ is a diffeomorphism when θ is real and near 0, and that it extends analytically to complex θ.

Lemma 5.3. *There exists $\delta > 0$ such that $\Phi_{\theta,T} : M \to M$ is a diffeomorphism when $\theta \in \mathbb{R}$, $e^\theta > 1 - \delta$. In addition, $(\det D\Phi_{\theta,T})^{1/2}$ extends analytically to the region*

$$\{\theta \in \mathbb{C} : \ |\operatorname{Im}\theta| < \frac{\pi}{2},\ e^\theta \notin (-\infty, 1-\delta)\}.$$

Now set

$$(U_{\theta,T} f)(a) = (\det D\Phi_{\theta,T})^{1/2} f(\Phi_{\theta,T}(a)). \tag{5.2}$$

Because of the simple geometric nature of the transformations U_θ and $U_{\theta,T}$, we may define the families of differential operators

$$\Delta_{\mathrm{rad},\theta} = U_\theta \Delta_{\mathrm{rad}} U_\theta^{-1}, \qquad \Delta_{\mathrm{rad},\theta,T} = U_{\theta,T}\Delta_{\mathrm{rad}} U_{\theta,T}^{-1},$$

without worrying about functional analytic issues of domain. These are W-invariant on $\mathfrak{a}$, with coefficients depending analytically on θ in the region $D = \{\theta : \ |\operatorname{Im}\theta| < \pi/2\} \subset \mathbb{C}$.

Indeed, we have already seen that $J_\theta^{1/2}$ extends to be analytic and nonvanishing on D. Since

$$U_\theta \Delta_{\mathrm{rad}} U_\theta^{-1} = J_\theta^{1/2} \Phi_\theta^* \Delta_{\mathrm{rad}} (\Phi_\theta^{-1})^* J_\theta^{-1/2},$$

we only need to consider $\Phi_\theta^*\Delta_{\mathrm{rad}}(\Phi_\theta^{-1})^*$. Now, the Φ_θ^*-conjugates of the principal part $\Delta_{\mathfrak{a}}$ (as well as the first order constant coefficient terms) continue to $\mathbb{C}\backslash\mathbb{R}^-$ (and even to a larger Riemann surface). For example, $\Phi_\theta^*\Delta_{\mathfrak{a}}(\Phi_\theta^{-1})^* = e^{-2\theta}\Delta_{\mathfrak{a}}$. However, the coefficients $\coth\alpha$ only continue up to $|\operatorname{Im}\theta| = \frac{\pi}{2}$, and genuine singularities appear in these continuations on this ray.

The coefficients of $\Delta_{\mathrm{rad},\theta}$ are thus smooth on $\mathfrak{a}$ when $|\operatorname{Im}\theta| < \frac{\pi}{2}$, but we also require information about their behaviour at $\partial\tilde{\mathfrak{a}}$.

Proposition 5.4. *If $\theta \in \mathbb{C}$ has $|\operatorname{Im}\theta| < \frac{\pi}{2}$, then Δ_θ is a (polyhomogeneous) conormal b-differential operator on $\overline{M}$. Its radial part $\Delta_{\mathrm{rad},\theta}$ is radially elliptic. The operators*

$$L_{b,\theta} = T_{b,\theta} + \Delta_{b,\mathrm{rad},\theta},\ \ b \in I^+,$$

on $L^2(S_b \times \Sigma^b; e^{2(\rho-\rho_b)}\, da_b\, dg_b)$, are product models for $\Delta_{\theta,\mathrm{rad}}$ in the sense that if $\chi_b \in \mathcal{C}^\infty(\hat{\mathfrak{a}})$ satisfies (i) and (iii) of Definition 3.5 then

$$E_{b,\theta}\chi_b = (\Delta_{\mathrm{rad},\theta} - L_{b,\theta})\chi_b \in x^\infty \operatorname{Diff}^1_{\mathrm{ss},o}(M).$$

Also, $\theta \to \Delta_{\mathrm{rad},\theta}$ is an analytic type-A family on $L^2(\tilde{\mathfrak{a}})^W$ with domain $H^2_{\mathrm{ss},o}(M)^K$.

Proof. The first part is easy from the explicit formula. We remark that $L_{b,\theta}$ is defined using the dilations on $S_b \times \Sigma^b$ and the Jacobian corresponding to the L^2-space

$$L^2(S_b \times \Sigma^b; e^{2(\rho-\rho_b)}\, da_b\, dg_b).$$

Thus, $\Delta_{b,\theta}$ is indeed the complex scaled Δ_b, defined by (5.1) with M replaced by Σ^b. Moreover, with $w = e^\theta$, $\tilde\rho = \rho - \rho_b$,

$$T_{b,\theta} = \mathcal{J}_\theta^{1/2}(w^{-2}\Delta_{S_b} + 2w^{-1}H_{\tilde\rho})\mathcal{J}_\theta^{-1/2}, \qquad \mathcal{J}_\theta = w^n e^{2(w-1)\tilde\rho},$$

so

$$T_{b,\theta} = e^{-\tilde\rho}(w^{-2}\Delta_{S_b} + |\tilde\rho|^2)e^{\tilde\rho}. \tag{5.3}$$

Now, since Δ_θ is radially elliptic, the domain of $\Delta_{\mathrm{rad},\theta}$ is $H^2_{\mathrm{ss},o}(M)^K$. For any $f \in H^2_{\mathrm{ss},o}(M)$, the map $\theta \mapsto \Delta_{\mathrm{rad},\theta} f \in L^2(M, dg)$ is strongly analytic, and this is what it means for $\Delta_{\mathrm{rad},\theta}$ to be an analytic family of type A. □

5.2. **Analytic vectors.** A general abstract theorem due to Nelson, cf. [19, Volume 2], uses the functional calculus to construct a dense set of analytic vectors for the generator of a group of unitary operators. We shall instead define an explicit subspace of analytic vectors $\mathbb{A}$, which is meant to demonstrate the essentially elementary nature of this result in our context. We ultimately wish to employ the operators $\Delta_{\mathrm{rad},\theta}$ for $\theta \in D = \{\theta : |\operatorname{Im}\theta| < \frac{\pi}{2}\}$, and using Nelson's theorem we could do this directly. A slight disadvantage with our more concrete approach is that this must be done in two steps now, first letting $\theta \in D' = \{\theta : |\operatorname{Im}\theta| < \frac{\pi}{4}\}$, and then extending to $\theta \in D$, but only a minor extra argument is needed for this.

The action of the Weyl group W extends naturally to $\mathfrak{a}_{\mathbb{C}}$. Define $\mathbb{A}$ to be the space of restrictions to $\mathfrak{a}$ of entire functions f on $\mathfrak{a}_{\mathbb{C}}$ which are W-invariant and which decay faster than any power of $e^{-|z|}$ in every cone $\{z \in \mathfrak{a}_{\mathbb{C}} : |\operatorname{Im} z| \le C|\operatorname{Re} z|\}$, $0 < C < 1$. In other words, denoting both the entire function and its restriction to $\mathfrak{a}$ by f, we have $f \in \mathbb{A}$ if, for every $0 < C < 1$ and $N > 0$,

$$\sup_{|\operatorname{Im} z| \le C|\operatorname{Re} z|} |f(z)| e^{N|z|} < +\infty.$$

Clearly, for any $\theta \in D'$ and $f \in \mathbb{A}$, $U_\theta f$ is rapidly decreasing on $\mathfrak{a}$.

Proposition 5.5. *For $\theta \in D'$, i.e. $|\operatorname{Im}\theta| < \frac{\pi}{4}$, $U_\theta \mathbb{A}$ is dense in $L^2(\mathfrak{a})^W$.*

Proof. Since $\mathcal{C}^0_c(\mathfrak{a})^W$ is dense in $L^2(\mathfrak{a})^W$ (with respect to the singular measure $dg_0 = \eta\, dx$ on $\mathfrak{a}$ – in this proof we use x for points in $\mathfrak{a}$), it suffices to show that any $f \in \mathcal{C}^0_c(\mathfrak{a})^W$ can be approximated by functions $f_t \in \mathbb{A}$. To this end, set

$$f_t(x) = c_n t^{-n/2} \int f(y) e^{-|x-y|^2/t}\, dy,$$

where $n = \dim \mathfrak{a}$ and c_n is chosen so that $\int f_t(x)\, dx = \int f(x)\, dx$ for all $t > 0$, i.e. so that $c_n t^{-n/2} e^{-|x|^2/t}$ is the Euclidean heat kernel. We claim first that $f_t \in \mathbb{A}$ when $t > 0$. Indeed, $f_t(x)$ is the restriction to $\mathfrak{a}$ of $f_t(z) = \int c_n t^{-n/2} e^{-(z-y)^2/t} f(y)\, dy$ and $\exp(-(z-y)^2)$ is entire in z and decreases faster than any power of $e^{-|z|}$ in $|\operatorname{Im} z| < C|\operatorname{Re} z|$ whenever $C < 1$, and this decay is preserved even after the integration over a compact set in y. Moreover, the action of W is by Euclidean isometries and hence commutes with the heat kernel, so each $f_t(x)$ is W-invariant. This proves the claim.

Now let us show that $U_\theta \mathbb{A}$ is dense in $L^2(\mathfrak{a})^W$ when $\theta \in D'$. For the case $\theta = 0$, note that for $f \in \mathcal{C}^0_c(\mathfrak{a})^W$, $e^{|x|^2} f_t$ is uniformly bounded when $t < 1$, and $\sup e^{|x|^2} |f(x) - f_t(x)| \to 0$ as $t \to 0$. Since $e^{-|x|^2} \in L^2(\mathfrak{a}; dg_0)^W$, we have $f_t \to f$ in in this space. In the general case, for any $\theta \in D'$, define

$$\tilde{f}_t(x) = c_n e^{n\theta} t^{-n/2} \int f(y) e^{-e^{2\theta}|x-y|^2/t}\, dy.$$

We must show that $\tilde{f}_t \to f$ in $L^2(\mathfrak{a})^W$ and $f_t \in U_\theta \mathbb{A}$. For the former, note that $\tilde{f}_t(x)$ is just the function $f_t(x)$ analytically continued to complex time $\tau = e^{-\theta} t$, and the same proof as above shows that $f_\tau \to f$ in L^2. Finally,

$$U_{-\theta} \tilde{f}_t(x) = c_n e^{n\theta/2} t^{-n/2} \int f(y) e^{-|x - e^\theta y|^2/t}\, dy$$

and as in the first part of the proof, this is certainly in $\mathbb{A}$. □

Corollary 5.6. *For* $|\operatorname{Im}\theta| < \frac{\pi}{4}$, $U_\theta\mathbb{A}$ *is dense in* $H^s_{\mathrm{ss},o}(M)$ *for any* $s \geq 0$.

Proof. Implicit in the definition of these Sobolev spaces, i.e. using radial ellipticity and the positivity of the Laplacian, cf. [15] for an explanation,

$$(\Delta_{\mathrm{rad}} + 1)^{s/2} : H^s_{\mathrm{ss},o}(M) \to H^0_{\mathrm{ss},o}(M) \equiv L^2(M, dg)^K \equiv L^2(\mathfrak{a}, dg_0)^W$$

is an isomorphism. Thus, $f_t \to f$ as $t \to 0$ in $H^s_{\mathrm{ss},o}(M)$ if and only if $(\Delta+1)^{s/2} f_t \to (\Delta+1)^{s/2} f$ in $L^2(\mathfrak{a}, dg_0)^W$. So given $f \in H^s_{\mathrm{ss},o}(M)$, let $k = (\Delta+1)^{s/2} f$. Since $\mathbb{A}$ is dense in $L^2(\mathfrak{a}; dg_0)^W$, there exists a family $k_t \in \mathbb{A}$ with $k_t \to k$ as $t \to 0$ in $L^2(\mathfrak{a}; dg_0)^W$. Now let $f_t = (\Delta+1)^{-s/2} k_t$ and note that $f_t \in \mathbb{A}$. Thus, $f_t \to f$ in $H^s_{\mathrm{ss},o}(M)$ as desired. □

For functions or distributions k which do not lie in $\mathbb{A}$, $U_\theta k$ may still have a continuation. For example, if $k = \delta_o$, the delta distribution at o, then using its homogeneity we see that for θ real, $U_\theta \delta_o = (\det D_o\Phi_\theta)^{-1/2}\delta_o$. Hence $U_\theta\delta_o$ extends to be analytic in θ (e.g. with values in some Sobolev space of sufficiently negative order), and so the Green function, $R(\lambda)\delta_o$ also extends via $\langle f, R(\lambda)\delta_o\rangle$ for $f \in \mathbb{A}$.

5.3. **The domain of continuation.** We now describe the Riemann surface $\widetilde{\mathcal{Y}}_{\pi/2}$ to which $R(\lambda)$ continues. We expect that $\widetilde{\mathcal{Y}}_{\pi/2}$ should be very simple, specifically either $\mathbb{C}$ or the Riemann surface for $\sqrt{z}$ or, at worst, for $\log z$, and in particular should be ramified at only one point. However, we only consider the continuation up to angle π ($\operatorname{Im}\theta = \pm\pi/2$), and in particular omit the ray where λ makes an angle of $\pm\pi$ with the spectral axis, and on which it is known that there exist poles of $R(\lambda)$ in many cases (e.g. on even dimensional hyperbolic spaces).

In addition, the N-body methods by themselves cannot rule out the existence of other poles in the nonphysical half-plane of $\sqrt{z}$. These poles are more serious than they might seem at first because in the inductive scheme, poles for the resolvent on spaces of rank less than n give rise to ramification points in the continuation for spaces of rank n. In the present paper we only describe the 'worst case scenario', and allow for the existence of these poles. We expect that the precise analysis of $\operatorname{Im}\theta \to \pm\pi/2$ will exclude their existence, see the discussion at the end of the last section.

Recall the symmetric space of lower rank, Σ^b, associated to S_b, $b \in I \setminus \{*\}$. Denote by $\mathcal{P}_{b,\theta}$ the pure point spectrum of $\Delta_{\Sigma^b,\mathrm{rad},\theta}$, and also assume that the set $\mathcal{T}_{b,\theta}$ of thresholds for $\Delta_{b,\mathrm{rad},\theta}$ has been defined inductively. Now define the set of thresholds for $\Delta_{\mathrm{rad},\theta}$, $\mathcal{T}_\theta$, by

$$\mathcal{T}_\theta = \bigcup_{b\neq *} \{|\rho - \rho_b|^2 + \gamma : \ \gamma \in \mathcal{P}_{b,\theta} \cup \mathcal{T}_{b,\theta}\}.$$

Note that for $b = 0$, Σ^b is a point, and so $\rho_b = 0$ and $\mathcal{P}_{0,\theta} = \{0\}$ for all θ; this means that we always have $|\rho|^2 \in \mathcal{T}_\theta$ for any θ. In addition, since $\rho - \rho_b \in S_b$ and $\rho_b \in S^b$ are orthogonal, this again contributes the value $|\rho - \rho_b|^2 + |\rho_b|^2 = |\rho|^2$ to $\mathcal{T}_\theta$. Presumably, $\mathcal{T}_\theta$ consists of the single element $|\rho|^2$, but this would rely on knowing that all spaces of rank less than n have no point spectrum and no thresholds except at $|\rho_b|^2$; in any case, this is true when $n = 2$.

We shall prove later, in Theorem 6.3, that as an operator on $L^2(\mathfrak{a}; dg_0)^W$,

$$\operatorname{spec}_{\mathrm{ess}}(\Delta_{\mathrm{rad},\theta}) = \{\gamma + e^{-2i\operatorname{Im}\theta}[0, +\infty) : \ \gamma \in \mathcal{T}(\theta)\} \tag{5.4}$$

when $|\operatorname{Im}\theta| < \pi/2$. In other words, every eigenvalue and threshold of the scaled radial Laplacian of each subsystem Σ^b contributes a ray to the essential spectrum of $\Delta_{\mathrm{rad},\theta}$ making an angle $-2\operatorname{Im}\theta$ with the positive real axis and emanating from that point. This ray is, in fact, the essential spectrum of the scaled 'tangential operator' $T_{b,\theta} = U_\theta^{-1}(\Delta_{S_b} + 2H_{\rho-\rho_b})U_\theta$. Granting this result, we now proceed with the rest of the complex scaling argument.

Normalize so that $\arg(z) \in (-2\pi, 0)$ for $z \in \mathbb{C} \setminus [0, +\infty)$, and let $\sqrt{z}$ be the branch of the square root function with $\operatorname{Im}\sqrt{z} < 0$ on $\mathbb{C} \setminus [0, +\infty)$. Let S be the Riemann surface of $\sqrt{\lambda - \lambda_0}$, with the ray with $\arg\sqrt{\lambda-\lambda_0} = \frac{\pi}{2}$ removed. The map

$$F : S \ni z = \sqrt{\lambda - \lambda_0} \mapsto \lambda = z^2 + \lambda_0$$

gives a double cover of $\mathbb{C} \setminus (-\infty, \lambda_0]$; the ray $(-\infty, \lambda_0)$ is only covered once. We call the part S_0 of S with $\operatorname{Im}\sqrt{\lambda-\lambda_0} < 0$, i.e. $\arg\sqrt{\lambda-\lambda_0} \in (-\pi, 0)$, the 'physical half-plane'.

We define Riemann surfaces $\mathcal{Y}_\beta$, $\beta \in [0, \pi/2]$, that are open subsets of S and such that $S_0 \subset \mathcal{Y}_\beta$. The part S_- of S with $\arg\sqrt{\lambda-\lambda_0} \in (-\pi/2, \pi/2)$ can be identified with $\mathbb{C} \setminus (-\infty, \lambda_0]$ via F. Then by definition

$$\begin{aligned}\mathcal{Y}_\beta \cap S_- \equiv &\{\lambda \in \mathbb{C} : \ \arg\sqrt{\lambda-\lambda_0} \in (-\pi/2, \beta)\} \\ &\qquad \setminus \{\gamma + e^{2i\beta}[0, +\infty) : \ \gamma \in \mathcal{T}(\beta)\}, \ \beta \in [0, \pi/2).\end{aligned} \tag{5.5}$$

Note that $\{\gamma + e^{2i\beta}[0, +\infty) : \ \gamma \in \mathcal{T}(\beta)\}$ is exactly the right hand side of (5.4) if we let $\operatorname{Im}\theta = -\beta$. With S_+ denoting the part of S with $\arg\sqrt{\lambda-\lambda_0} \in (-3\pi/2, -\pi/2)$, we define

$$\begin{aligned}\mathcal{Y}_\beta \cap S_+ \equiv &\{\lambda \in \mathbb{C} : \ \arg\sqrt{\lambda-\lambda_0} \in (-\pi-\beta, -\pi/2)\} \\ &\qquad \setminus \{\gamma + e^{-2i\beta}[0, +\infty) : \ \gamma \in \mathcal{T}(\beta)\}, \ \beta \in [0, \pi/2).\end{aligned}$$

Note that with this definition, $\mathcal{Y}_0$ is the 'physical half plane' S_0.

Remark 5.7. Although each $\mathcal{Y}_\beta$ can be considered as a subset of S, it is important to realize that even in the overlap of these regions for different values of β, the $\mathcal{Y}_\beta$ should not be identified with each other. Rather, two points $p \in \mathcal{Y}_\beta$ and $q \in \mathcal{Y}_\gamma$ with $\gamma \leq \beta$ with the same image λ' in S_-, say, should only be identified if

$$\lambda' \notin \{\gamma + e^{2i\theta}[0, +\infty) : \ \gamma \in \mathcal{T}(\theta), \ \theta \in [\gamma, \beta]\}.$$

An equivalent formulation would be that the two points should be identified if there is a path in S_- connecting λ' to 'physical region' $\arg\sqrt{\lambda-\lambda_0} \in (-\pi/2, 0)$ which stays entirely in the intersection of $S_- \cap \mathcal{Y}_\beta$ and $S_- \cap \mathcal{Y}_\gamma$.

For this reason we make the following definition.

Definition 5.8. For $\beta \in (0, \pi/2]$, we define $\widetilde{\mathcal{Y}}_\beta$ as the disjoint union of $\mathcal{Y}_\gamma$, $\gamma \in [0, \beta)$, modulo the equivalence relation described above. We define the topology of $\widetilde{\mathcal{Y}}_\beta$ by requiring that open subsets of $\mathcal{Y}_\gamma$ to be open in $\widetilde{\mathcal{Y}}_\beta$, and taking these as a base for the topology of $\widetilde{\mathcal{Y}}_\beta$ as γ runs over $[0, \beta)$. Letting the $\mathcal{Y}_\gamma$ be coordinate charts, we make $\widetilde{\mathcal{Y}}_\beta$ into a Riemann surface.

Remark 5.9. In this definition, if $\beta < \frac{\pi}{2}$, we could replace $\gamma \in [0, \beta)$ by $\gamma \in [0, \beta]$; the resulting Riemann surface would be the same.

Denote by $R(\lambda,\theta)$ the operator $(\Delta_{\mathrm{rad},\theta}-\lambda)^{-1}$. To be definite, we consider only the analytic continuation of $R(\lambda)=R(\lambda,0)$ from the lower right quadrant $\mathrm{Im}(\lambda-\lambda_0)<0$ through the ray $(\lambda_0,+\infty)$), i.e. to $S_-\cap\mathcal{Y}_\beta$; the continuation from $\mathrm{Im}(\lambda-\lambda_0)>0$ is handled nearly identically.

The main point, roughly speaking, is that when $-\frac{\pi}{2}<\mathrm{Im}\,\theta<0$, $\Delta_\theta-\lambda$ is a holomorphic family of operators (in λ) with values in the space of radially elliptic operators on M. Thus $R(\lambda,\theta)$ is meromorphic in λ outside $\mathrm{spec}_{\mathrm{ess}}(\Delta_{\mathrm{rad},\theta})$ with values in bounded operators on $L^2(\mathfrak{a};dg_0)^W$. This family has only finite rank poles, and these are the poles of the continuation of $R(\lambda)_{\mathrm{rad}}$ in $\mathcal{Y}_\beta\cap S_-$ if we choose θ so that $\beta=-\mathrm{Im}\,\theta<\frac{\pi}{2}$.

5.4. **Continuation of the resolvent.** We finally indicate the proof of the analytic continuation of the resolvent, which is simply an application of the theorem of Aguilar-Balslev-Combes in our setting.

Theorem. ([9, Theorem 16.4]) *Suppose that U_θ and $\mathbb{A}$ satisfy the hypotheses (i)-(iii) listed in the beginning of §4, and that Δ_θ is a type-A analytic family in the strip $D'=\{\theta:\ |\mathrm{Im}\,\theta|<\frac{\pi}{4}\}$, and (5.4) holds for $\theta\in D$. Then*

- (i) *For $f,g\in\mathbb{A}$, $\beta<\frac{\pi}{4}$, the function $\langle f,R(\lambda)_{\mathrm{rad}}g\rangle$ has a meromorphic continuation to $\mathcal{Y}_\beta$.*
- (ii) *The poles of the continuation of $\langle f,R(\lambda)g\rangle$ to $\mathcal{Y}_\beta$, $\beta<\frac{\pi}{4}$, are the eigenvalues of $\Delta_{\mathrm{rad},\beta}$.*
- (iii) *The poles are independent of the choice of U_θ in the sense that if U'_θ and $\mathbb{A}'$ also satisfy (i)-(iii) and if $\mathbb{A}\cap\mathbb{A}'$ is dense, then the eigenvalues of $U'_\theta\Delta_{\mathrm{rad}}(U'_\theta)^{-1}$ are the same as those of $\Delta_{\mathrm{rad},\theta}$.*

All of the hypotheses have already been discussed and verified. We shall briefly outline the proof of the first part since the idea is simple. To relate $R(\lambda,\theta)$ and $R(\lambda)$, fix $\epsilon>0$, and suppose that

$$\theta\in\Omega_\epsilon=\{-\epsilon<\mathrm{Im}\,\theta<\frac{\pi}{4}\}\qquad\text{and}\qquad\arg(\lambda-\lambda_0)\in(-\pi,-\epsilon).$$

When θ is real, U_θ is unitary and so

$$\langle f,R(\lambda)g\rangle=\langle U_{\bar\theta}f,(U_\theta R(\lambda)U_\theta^{-1})U_\theta g\rangle=\langle U_{\bar\theta}f,R(\lambda,\theta)U_\theta g\rangle \tag{5.6}$$

since $U_\theta R(\lambda)U_\theta^{-1}=R(\lambda,\theta)$. The left side of this equation is independent of θ, while the expression on the (far) right is analytic in θ on Ω_ϵ, hence is also constant on this domain. This holds when $\arg(\lambda-\lambda_0)\in(-\pi,-\epsilon)$.

To extend $\langle f,R(\lambda)g\rangle$ to $\mathcal{Y}_\beta$, take θ with $\mathrm{Im}\,\theta=-\beta$. Then for $\lambda\in\mathbb{C}$ with $\mathrm{Im}(\lambda-\lambda_0)<0$, $\langle f,R(\lambda)g\rangle$ is given by the right hand side of (5.6). But this right side is analytic in λ away from the spectrum of $\Delta_{\mathrm{rad},\theta}$, and meromorphic away from its essential spectrum, hence is meromorphic on $\mathcal{Y}_\beta$, as claimed.

This continuation is clearly independent of the choice of θ with $-\mathrm{Im}\,\theta=\beta$ since any such continuation is a meromorphic function of λ that agrees with a given function on an open set. In addition, the continuation is independent of β in the sense that if $p\in\mathcal{Y}_\beta$ and $q\in\mathcal{Y}_\gamma$ are identified in the sense of Remark 5.7, so there is a path connecting them to the physical region that does not intersect the cuts in either $\mathcal{Y}_\beta$ or in $\mathcal{Y}_\gamma$, then $\langle f,R(\lambda)g\rangle$ is the same whether the β or γ is used to define it.

Note that this does not yet quite say that $R(\lambda)\delta_o$ continues as a distribution, since that would require that the right hand side of (5.6) be defined for any $f\in\mathcal{C}_c^\infty(\mathfrak{a})^W$, while for most f, $U_\theta f$ does *not* have an analytic extension from the real axis. This is where we require the deformed group of unitary operators, $U_{\theta,T}$, defined in (5.2). Recall that the associated diffeomorphisms $\Phi_{\theta,T}$ fixes $B_T(o)$ pointwise and equals Φ_θ when $|a|$ is sufficiently large. We use precisely the same arguments as above to establish the density of $U_{\theta,T}\mathbb{A}$. Hence by the uniqueness part of the Aguilar-Balslev-Combes theorem, the induced analytic extensions agree with one another no matter the value of T, and also agree with the extension associated to U_θ. But if $f \in \mathcal{C}_c^\infty(B_T(o))^W$, then $U_{\theta,T}f = f$ and so $U_{\theta,T}f = f$ has an analytic extension to $\theta \in \mathbb{C}$. Arguing as before, the formula

$$\langle f, R(\lambda)\delta_o\rangle = \langle U_{\bar\theta,T}f, R(\lambda,\theta,T)U_{\theta,T}\delta_o\rangle = \langle f, R(\lambda,\theta,T)\delta_o\rangle \tag{5.7}$$

shows that $R(\lambda)\delta_o$ does indeed extend analytically as a distribution to $\mathcal{Y}_\beta$, $\beta \in (0,\frac{\pi}{4})$, since the right hand side has this property.

Although we have only constructed a subset $\mathbb{A} \subset L^2(\mathfrak{a}; dg_0)^W$ for which $U_\theta\mathbb{A}$ is dense in $L^2(\mathfrak{a}; dg_0)^W$ when $|\operatorname{Im}\theta| < \pi/4$, we can still continue $R(\lambda)$ to $\widetilde{\mathcal{Y}}_{\pi/2}$, for which the formula (5.6) requires larger $\operatorname{Im}\theta$.

Theorem (Theorem 1.1). *The Green function $G_o(\lambda)$ continues meromorphically to $\widetilde{\mathcal{Y}}_{\pi/2}$ as a distribution.*

Proof. We have shown that the hypotheses of the Aguilar-Balslev-Combes theorem are satisfied for $D' = \{\theta : \ |\operatorname{Im}\theta| < \frac{\pi}{4}\}$ (for either U_θ or $U_{\theta,T}$) (except for the proof of (5.4)). Hence $R(\lambda)$ continues meromorphically to $\mathcal{Y}_\beta$, $\beta \in (0,\pi/4)$, in the precise sense of the theorem. In particular, $G_o(\lambda)$ continues meromorphically to $\mathcal{Y}_\beta$ as a distribution. However, at first we ignore the continuation itself, i.e. restrict to λ with $\arg\sqrt{\lambda-\lambda_0} \in (-\pi/2, 0)$, and extend the scaling argument instead.

Namely, we use the semigroup property $U_\theta U_{\theta'} = U_{\theta+\theta'}$, which implies the analogue of (5.6):

$$\langle f, R(\lambda,\theta')g\rangle = \langle U_{\bar\theta}f, R(\lambda,\theta+\theta')U_\theta g\rangle \tag{5.8}$$

for $f, g \in \mathbb{A}$, $|\operatorname{Im}\theta| < \frac{\pi}{4}$, $\arg\sqrt{\lambda-\lambda_0} \in (-\pi/2, 0)$. Hence $U_\theta R(\lambda,\theta')U_\theta^{-1} = R(\lambda,\theta+\theta')$ for $\theta \in \mathbb{R}$, and so (5.8) gives the continuation of $R(\lambda,\theta')$ to $\lambda \in \mathcal{Y}_{-\operatorname{Im}\theta'-\operatorname{Im}\theta}$. For $\beta \in (0,\pi/2)$, we may take θ, θ' with $\operatorname{Im}\theta = \operatorname{Im}\theta' = -\beta/2$, so we conclude that $R(\lambda)$ continues analytically to $\mathcal{Y}_\beta$.

This also gives the extension of $R(\lambda)\delta_o$ to $\mathcal{Y}_\beta$ as a distribution. Indeed, this extension exists in $\mathcal{D}'(B_T(o))$ for any $T > 0$, and the density of $\mathbb{A}$ implies that these extensions are all the same.

Finally, by the very definition of $\widetilde{\mathcal{Y}}_{\pi/2}$, the analytic continuation of $G_o(\lambda) = R(\lambda)\delta_o$ to $\mathcal{Y}_\beta$ for every $\beta \in (0,\pi/2)$ gives the desired analytic continuation to $\widetilde{\mathcal{Y}}_{\pi/2}$. □

Remark 5.10. We emphasize that although the analytic extension to $\mathcal{Y}_\beta$, $\beta \in [\pi/4, \pi/2)$ is defined in two steps, the analytic extension of δ_o as a distribution on $B_T(o)$ can be done at once. Indeed, both $U_{\theta,T}\delta_o$ and $U_{\theta,T}f$, $f \in \mathcal{C}_c^\infty(B_T(o))$, have an analytic extension to $\{\theta : \ |\operatorname{Im}\theta| < \pi/2\}$, so (5.7) defines the extension (in $\mathcal{C}^{-\infty}(B_T(o))$) of $R(\lambda)\delta_o$ directly in the region $\mathcal{Y}_\beta$, $\beta \in (0,\pi/2)$.

6. The parametrix construction

Our final goal is to identify the essential spectrum of $\Delta_{\mathrm{rad},\theta}$ when $|\operatorname{Im}\theta| < \pi/2$. As usual, the strategy is to construct a parametrix for the scaled resolvent

$(\Delta_{\mathrm{rad},\theta} - \lambda)^{-1}$ with compact remainder when λ is outside the putative essential spectrum. We shall approach this in a series of steps. The procedure is inductive, and the parametrix is built up from the resolvents of the scaled model operators $L_{b,\theta} = T_{b,\theta} + \Delta_{\Sigma^b,\mathrm{rad},\theta}$, $b \in I$, localized to neighbourhoods of $S_b \times \{0\} \subset S_b \times S^b$ (for $b = *$, $L_{b,\theta} = \Delta_{\mathrm{rad},\theta}$ and we localize to a compact neighbourhood of $0 \in \mathfrak{a}$). In the first step, we use the 'softest' form of this induction, employing only radial ellipticity, to obtain an exact inverse to $\Delta_{\mathrm{rad},\theta} - \lambda$ when λ is sufficiently large and lies outside any small cone surrounding the essential spectrum. We also obtain decay estimates for the norm of the resolvent as $|\lambda| \to \infty$. The point is that we are able to get a parametrix with remainder which has small norm, which can then be inverted away using a Neumann series. This involves the use either of the associated semiclassical calculus or, perhaps more familiarly, a pseudodifferential calculus with spectral parameter, as described for example in [20]; see also [25] where this is used in the N-body setting. These decay estimates are necessary in the next step, where we use the convolution formula for the resolvent on a product space from [14] to describe the resolvents $(L_{b,\theta} - \lambda)^{-1}$ in terms of the resolvents for $T_{b,\theta}$ and $\Delta_{\Sigma^b,\mathrm{rad},\theta}$; here we use the induction hypothesis, specifically the estimates from the first step, for the latter factor. A slight technical twist is that we need to modify this formula to handle sums of nonselfadjoint operators. This would follow from a more general abstract theorem (Ichinose's lemma), but we also indicate a direct proof. In the third and final step we use the resolvents of the model operators obtained in the previous step to obtain a parametrix for $(\Delta_{\mathrm{rad},\theta} - \lambda)^{-1}$ with a *compact* remainder, for all λ outside the essential spectrum. After this we can finish the whole construction by applying the analytic Fredholm theorem.

Step 1: The parametrix for large spectral parameter

As described above, the first task is to construct and obtain estimates on the resolvent $(\Delta_{\mathrm{rad},\theta} - \lambda)^{-1}$ when λ tends to infinity and remains outside some sector. More precisely, we show that for any $\epsilon > 0$, and $R = R_\epsilon > 0$ sufficiently large, depending on ϵ,

$$\mathrm{spec}(\Delta_{\mathrm{rad},\theta}) \cap \{|\lambda| > R\} \subset e^{-2i[\mathrm{Im}\,\theta - \epsilon, \mathrm{Im}\,\theta + \epsilon]}[0, +\infty) \cap \{|\lambda| > R\} := D^c_{R,\epsilon},$$

and for λ large and outside this latter set we estimate the norm of $(\Delta_{\mathrm{rad},\theta} - \lambda)^{-1}$ on $L^2(M)^K$ in terms of powers of $1/|\lambda|$. This is proved by constructing a parametrix with error term which tends to zero in operator norm as $\lambda \to \infty$, and which then be inverted away. This step is 'soft' inasmuch as we only use radial ellipticity in this argument, but we emphasize that this error term is small, but not necessarily compact.

One could proceed rather abstractly at this stage by showing that $\Delta_{\mathrm{rad},\theta}$ is m-sectorial, cf. [19, Volume II, Section VIII.6]. This would involve considering the quadratic form $\langle \phi, \Delta_{\mathrm{rad},\theta}\phi \rangle$ for $\phi \in \mathcal{C}^\infty_c(M)^K$. The point here is that the difference between $\Delta_{\mathfrak{a},\theta}$ and $\Delta_{\mathrm{rad},\theta}$ is a first order differential operator, and the form corresponding to this difference can be estimated via Cauchy-Schwartz. However, the fact that we must use a nontrivial measure on $\mathfrak{a}$ because of the identification $L^2(M)^K \cong L^2(\mathfrak{a}, dg)^W$ makes this not entirely trivial.

However, in keeping with the other steps, we construct the parametrix by piecing together the simplest of parametrices for the model operators $L_{b,\theta}$ using a $(W, \overline{\mathfrak{a}})$-adapted partition of unity, maintaining control on large λ behaviour.

Proposition 6.1. *For any $\epsilon > 0$ there exist $R, C > 0$ such that when $|\lambda| > R$ and $|\arg\lambda + 2\operatorname{Im}\theta| > \epsilon$, we have*

$$R(\lambda,\theta) = (\Delta_{\mathrm{rad},\theta} - \lambda)^{-1} \in \mathcal{B}(L^2(M)^K),$$
$$\|R(\lambda,\theta)\|_{\mathcal{B}(L^2(M)^K)} \leq \frac{C}{|\lambda|}.$$

Proof. Recall that, for any $b \in I$, $L_{b,\theta} - \lambda = T_{b,\theta} + \Delta_{b,\mathrm{rad},\theta} - \lambda$ is an operator on $S_b \times \Sigma^b$ which is constant coefficient on the first factor and radial on the second; moreover, we are only interested in its restriction to a fixed bounded neighbourhood in Σ^b. For λ outside this sector, this is an elliptic element of the pseudodifferential calculus with large spectral parameter (satisfying uniform estimates in the S_b factor), as defined in [20]. Choose two different sets of cutoffs, $\{\phi_b\}$ and $\{\psi_b\}$, $b \in I$, each satisfying (i)-(iii) of Definition 3.7, and such that ψ_b is identically 1 on a neighbourhood of $\operatorname{supp}\phi_b$ and $\operatorname{supp}\psi_b$ is sufficiently close to $\overline{S_b}$; the smallness of the support ensures that $\Delta_{\Sigma^b,\theta}$ is elliptic on it. There exists a parametrix in this calculus, $G_{b,\theta}(\lambda)$, which we may as well assume is K^b-invariant (by averaging it over K^b), which is supported near $\operatorname{supp}\psi_b$. This satisfies the analogues of the bounds in the statement of this proposition, and in addition,

$$(L_{b,\theta} - \lambda)G_{b,\theta}(\lambda)\phi_b = \phi_b + F_{b,\theta}(\lambda),$$

where $\|F_{b,\theta}(\lambda)\|_{\mathcal{B}(L^2(M)^K)} \leq C_{N,\epsilon}/|\lambda|^N$ for any $N, \epsilon > 0$, by virtue of the properties of residual elements in this large parameter calculus. Finally, define

$$G_\theta(\lambda) = \sum_b \psi_b G_{b,\theta}(\lambda)\phi_b.$$

We have

$$(\Delta_{\mathrm{rad},\theta} - \lambda)G_\theta(\lambda) = \mathrm{Id} + \sum_b \left([\Delta_{\mathrm{rad},\theta}, \psi_b]G_{b,\theta}(\lambda)\phi_b + \psi_b F_{b,\theta}(\lambda)\right) = \mathrm{Id} + F_\theta(\lambda).$$

Since $\operatorname{supp}[\Delta_{\mathrm{rad},\theta}, \psi_b]$ is disjoint from $\operatorname{supp}\phi_b$, this error term also satisfies

$$\|F_\theta(\lambda)\|_{\mathcal{B}(L^2(M)^K)} \leq \frac{C_N}{|\lambda|^N}$$

for any $N, \epsilon > 0$. Thus $\mathrm{Id} + F_\theta(\lambda)$ is invertible when $|\lambda| > R$ (still outside this sector), so

$$(\Delta_{\mathrm{rad},\theta} - \lambda)G_\theta(\lambda)(\mathrm{Id} + F_\theta(\lambda))^{-1} = \mathrm{Id},$$

and standard arguments also show that this is a left inverse too. This means that

$$(\Delta_{\mathrm{rad},\theta} - \lambda)^{-1} = R(\lambda,\theta) = G_\theta(\lambda)(\mathrm{Id} + F_\theta(\lambda))^{-1}.$$

The estimates for $R(\lambda,\theta)$ follow directly from those for $G_{b,\theta}(\lambda)$. □

Step 2: Resolvents of the model operators

We now use the convolution formula from [14] and the decay estimates obtained in the previous step to express the resolvent for each model operator

$$L_{b,\theta} = T_{b,\theta} + \Delta_{\Sigma^b,\theta} \tag{6.1}$$

in terms of the resolvents of the two summands. We assume now that $b \neq *$, since the analysis of $L_{*,\theta} = \Delta_{\mathrm{rad},\theta}$ is what we are ultimately trying to understand. Note also the other extreme case $b = 0$, where $L_{0,\theta} = (\Delta_{\mathfrak{a}})_\theta = e^{-2\theta}\Delta_{\mathfrak{a}}$.

The first summand is a constant coefficient operator on S_b which is the rescaling of

$$T_b = \Delta_{S_b} + 2(H_\rho - H_{\rho_b}).$$

Recall that if M_f is the operator of multiplication by a function $f > 0$, then

$$M_f : L^2(S_b, f^2\, da_b) \to L^2(S_b, da_b)$$

is a unitary isomorphism. Thus choosing $f = e^{\rho-\rho_b}$, then we see that T_b acting on $L^2(S_b, e^{2(\rho-\rho_b)}\, da_b)$ is unitarily equivalent to

$$\tilde{T}_b = f^{-1}(\Delta_{S_b} + 2H_{\rho-\rho_b})f = \Delta_{S_b} + (\rho-\rho_b)\cdot(\rho-\rho_b) = \Delta_{S_b} + |\rho-\rho_b|^2, \tag{6.2}$$

acting on $L^2(S_b, da_b)$, and correspondingly, using the same f, see (5.3), $T_{b,\theta}$ is unitarily equivalent to

$$\tilde{T}_{b,\theta} = \Delta_{S_b,\theta} + |\rho-\rho_b|^2,$$

also on $L^2(S_b, da_b)$. In particular, since $\Delta_{S_b,\theta} = e^{-2\theta}\Delta_{S_b}$, it follows immediately that

$$\operatorname{spec}(T_{b,\theta}) = |\rho-\rho_b|^2 + e^{-2i\operatorname{Im}\theta}[0,+\infty). \tag{6.3}$$

In addition, from the Fourier transform representation of this operator we deduce that

$$\|(T_{b,\theta}-\lambda)^{-1}\| \le C/|\lambda| \tag{6.4}$$

as $\lambda \to \infty$ away from $D^c_{R,\epsilon}$.

Since the rank of Σ^b is strictly less than n, the spectrum of the other summand in (6.1) is understood by induction. Because these rescaled operators are not self-adjoint, it is not completely trivial that the spectrum of $L_{b,\theta}$ is the sum of spectra of the two operators on the right. This follows from an abstract lemma due to Ichinose, cf. [19, Volume IV, Section XIII.9, Corollary 2], but also follows directly from the existence of the resolvent when λ is outside the sum of these two spectra:

Corollary 6.2. *For any $b \in I\backslash\{*\}$, as an operator on $L^2(\Sigma^b \times S_b, e^{2(\rho-\rho_b)}\, da_b\, dg_b)$),*

$$\operatorname{spec}(L_{b,\theta}) = \{\lambda' + \lambda'' :\ \lambda' \in \operatorname{spec}(\Delta_{\Sigma^b,\theta}),\ \lambda'' \in |\rho-\rho_b|^2 + e^{-2i\operatorname{Im}\theta}[0,+\infty)\}. \tag{6.5}$$

In particular, outside this set,

$$R_{b,\theta}(\lambda) = (L_{b,\theta}-\lambda)^{-1} \in \mathcal{B}(L^2(\Sigma^b \times S_b, e^{2(\rho-\rho_b)}\, da_b\, dg_b)).$$

Proof. The convolution formula states that

$$R_{b,\theta}(\lambda) = \frac{1}{2\pi i}\int_\gamma (\Delta_{\Sigma^b,\theta}-\mu)^{-1} \otimes (T_{b,\theta}-(\lambda-\mu))^{-1}\, d\mu, \tag{6.6}$$

where γ is a path in $\mathbb{C}$ which avoids $\operatorname{spec}(\Delta_{\Sigma^b,\theta})$ and $\lambda - \operatorname{spec}(T_{b,\theta})$, and which diverges linearly from these rays. The decay estimates

$$\|(\Delta_{\Sigma^b,\theta}-\mu)^{-1}\| \le |\operatorname{Im}\mu|^{-1},\ \|(T_{b,\theta}-(\lambda-\mu))^{-1}\| \le |\operatorname{Im}(\lambda-\mu)|^{-1}$$

from Proposition 6.1 and (6.4) show that this integral converges as a bounded operator. Note that the operator defined by this integral agrees with the scaled resolvent follows by first varying θ while keeping γ fixed, and then everywhere outside the set (6.5) by virtue of the analytic dependence on λ. □

Step 3: The parametrix with compact remainder

We now prove the main

Theorem 6.3. *The operator* $\Delta_{\mathrm{rad},\theta}$ *has essential spectrum*

$$\operatorname{ess\,spec}(\Delta_{\mathrm{rad},\theta}) = \bigcup_{b\in I^+\setminus\{*\}} \operatorname{spec}(L_{b,\theta}). \tag{6.7}$$

The map

$$\lambda \mapsto R(\lambda) = (\Delta_{\mathrm{rad},\theta} - \lambda)^{-1}$$

is meromorphic on $\mathbb{C}\setminus \cup_{b\neq *}\operatorname{spec}(L_{b,\theta})$ *with residues of finite rank.*

The inclusion of the set on the right side of (6.7) into the set on the left is immediate because $\Delta_{\mathrm{rad},\theta}$ is well approximated by each of the $L_{b,\theta}$ in appropriate neighbourhoods of infinity. To prove the inclusion of the set on the left into the set on the right, it suffices to prove that when λ is outside the spectrum of $L_{b,\theta}$ for every $b \neq *$, then there is a parametrix for the operator $(\Delta_{\mathrm{rad},\theta} - \lambda)^{-1}$ with compact remainder.

As before, choose a $(W,\hat{\mathfrak{a}})$-adapted partition of unity $\{\phi_b\}$, $b \in I^+$, on the geodesic compactification $\hat{\mathfrak{a}}$ of $\mathfrak{a}$, and let $\{\psi_b\}$, $b\in I^+$, be a corresponding collection of cutoff functions on $\hat{\mathfrak{a}}$, so $\psi_b \in \mathcal{C}^\infty(\hat{\mathfrak{a}})$ satisfies (i)-(iii) of Definition 3.5 and such that ψ_b is identically 1 in a neighbourhood of $\operatorname{supp}\phi_b$.

Denote by $\pi : M \to \overline{\mathcal{C}^+}$ and $\pi^b : \Sigma^b \to S^b_+$ the projections induced by the Cartan decompositions on M and Σ^b. On a neighbourhood U_b of $\operatorname{supp}\psi_b$, $L^2(\pi^{-1}(U_b), dg)^K$ may be identified with $L^2(\pi_b^{-1}(U_b), e^{2(\rho-\rho_b)}\,da_b\,dg_b)^{K^b}$.

We assume, by induction, that the spectrum of $L_{b,\theta}$ is known for every $b \in I^+\setminus\{*\}$. As above, for every such b let $R_{b,\theta}(\lambda) = (L_{b,\theta}-\lambda)^{-1}$ for $\lambda \notin \operatorname{spec}(L_{b,\theta})$. When $b = *$, let $R_{*,\theta}$ denote an ordinary K-invariant parametrix for $\Delta_{\mathrm{rad},\theta}$ on some large ball in $\mathfrak{a}$. The restriction of every $\psi_b R_{b,\theta}(\lambda)\phi_b$ to K^b-invariant functions may be regarded as acting on K-invariant functions on M, and with this identification we define the parametrix

$$P_\theta(\lambda) = \sum_b \psi_b R_{b,\theta}(\lambda)\phi_b.$$

Proposition 6.4. *For any* $k,l,r,s \in \mathbb{R}$ *and* $\lambda \notin \operatorname{spec}(L_{b,\theta})$, *and* $\hat{x}$ *a defining function for* $\partial\hat{\mathfrak{a}}$,

$$R_{b,\theta}(\lambda) : \hat{x}^k H^s_{\mathrm{ss},o}(M) \longrightarrow \hat{x}^k H^{s+2}_{\mathrm{ss},o}(M), \tag{6.8}$$

is bounded; moreover, if $\chi,\phi \in \mathcal{C}^\infty(\hat{\mathfrak{a}})^W$ *have disjoint support, then*

$$\chi R_{b,\theta}(\lambda)\phi : \hat{x}^k H^s_{\mathrm{ss},o}(M) \to \hat{x}^l H^r_{\mathrm{ss},o}(M). \tag{6.9}$$

Proof. The argument below does not depend on θ at all, so we suppress the scaling in the already cumbersome notation. Also, assume $b \in I^+\setminus\{*\}$, since the result is straightforward when $b = *$.

We first show that (6.8) implies (6.9). In fact, since the supports of χ and ϕ are disjoint,

$$\chi R_b(\lambda)\phi = [\chi, R_b(\lambda)]\phi = R_b(\lambda)[L_b,\chi]R_b(\lambda)\phi.$$

Certainly $[L_b,\chi] \in \hat{x}\operatorname{Diff}^1_{\mathrm{ss},o}(S_b\times\Sigma^b)$ by the Proposition 4.7, hence is bounded as a map $\hat{x}^k H^{s+2}_{\mathrm{ss},o}(M) \to \hat{x}^{k+1}H^{s+1}_{\mathrm{ss},o}(M)$ due to Proposition 4.6. Using (6.8), we deduce that

$$\chi R_b(\lambda)\phi : \hat{x}^k H^s_{\mathrm{ss},o}(M) \to \hat{x}^{k+1}H^{s+3}_{\mathrm{ss},o}(M);$$

iterating this proves the claim.

Let us now prove (6.8). The case $k = 0$ follows from elliptic regularity and the definition of the spaces $H^s_{\mathrm{ss},o}(M)$. For general k, we must show that

$$\hat{x}^k R_b(\lambda)\hat{x}^{-k} : H^s_{\mathrm{ss},o}(M) \longrightarrow H^{s+2}_{\mathrm{ss},o}(M).$$

Assume that $k > 0$ since the case $k < 0$ then follows by applying the argument below to the adjoint. Using the identity

$$[R_b(\lambda), \hat{x}^{-k}] = R_b(\lambda)[\hat{x}^{-k}, L_b]R_b(\lambda),$$

we have

$$\hat{x}^k R_b(\lambda)\hat{x}^{-k} = R_b(\lambda) + \hat{x}^k[R_b(\lambda), \hat{x}^{-k}] = R_b(\lambda) + \hat{x}^k R_b(\lambda)[\hat{x}^{-k}, L_b]R_b(\lambda).$$

Obviously the first term on the right is bounded from $H^s_{\mathrm{ss},o}(M)$ to $H^{s+2}_{\mathrm{ss},o}(M)$. Next, $[\hat{x}^{-k}, L_b] : H^r_{\mathrm{ss},o}(M) \to H^{r-1}_{\mathrm{ss},o}(M)$ is bounded provided $0 \leq k \leq 1$. Applying this with $r = s + 2$, and using that multiplication by $\hat{x}^k$ is bounded on $H^s_{\mathrm{ss},o}(M)$, we see that the second term on the right is bounded from $H^s_{\mathrm{ss},o}(M)$ to $H^{s+3}_{\mathrm{ss},o}(M)$, so altogether $R_b(\lambda) : \hat{x}^k H^s_{\mathrm{ss},o}(M) \to \hat{x}^k H^{s+2}_{\mathrm{ss},o}(M)$ is bounded when $|k| \leq 1$.

In general, if it is known that $R_b(\lambda) : \hat{x}^l H^s_{\mathrm{ss},o}(M) \to \hat{x}^l H^{s+2}_{\mathrm{ss},o}(M)$ is bounded for some $l > 0$, then the identity

$$\hat{x}^{k-l} R_b(\lambda)\hat{x}^{-k+l} = R_b(\lambda) + \hat{x}^{k-l} R_b(\lambda)[\hat{x}^{-k+l}, L_b]R_b(\lambda)$$

shows that it is true for any k with $l < k \leq l + 1$. (This uses the boundedness of $[\hat{x}^{k-l}, L_b] : \hat{x}^l H^{s+2}_{\mathrm{ss},o}(M) \to \hat{x}^l H^{s+1}_{\mathrm{ss},o}(M)$.) This proves the result for all k. □

Proposition 6.5. *For* $\lambda \in \mathbb{C} \setminus \cup_{b\neq *} \operatorname{spec}(L_{b,\theta})$,

$$P_\theta(\lambda)(\Delta_{\mathrm{rad},\theta} - \lambda) - \mathrm{Id}, (\Delta_{\mathrm{rad},\theta} - \lambda)P_\theta(\lambda) - \mathrm{Id} : \hat{x}^k H^s_{\mathrm{ss},o}(M) \to \hat{x}^l H^{s+1}_{\mathrm{ss},o}(M),$$

for any $s, k, l \in \mathbb{R}$.

Proof. Again θ plays no role, so we drop it from the notation.

For λ in the specified domain, each $R_b(\lambda)$ is bounded on $L^2(M)^K$, by Corollary 6.2. Now

$$(\Delta - \lambda)P(\lambda) = \sum_{b\in I^+} (\Delta - \lambda)\psi_b R_b(\lambda)\phi_b.$$

On $\operatorname{supp}\psi_b$, $b \neq *$, $\Delta = L_b + E_b$. Here

$$E_b\psi_b : \hat{x}^k H^s_{\mathrm{ss},o}(M) \to \hat{x}^l H^{s-1}_{\mathrm{ss},o}(M) \tag{6.10}$$

for any k, l, s since $E_b\psi_b \in x^\infty \operatorname{Diff}^1_{\mathrm{ss},o}(M)$ by Lemma 2.3 (and Proposition 5.4 for $\theta \notin \mathbb{R}$). Hence

$$(\Delta - \lambda)P(\lambda) = \sum_{b\neq *} E_b\psi_b R_b(\lambda)\phi_b + \sum_b [L_b, \psi_b]R_b(\lambda)\phi_b + \sum_b \psi_b(L_b - \lambda)R_b(\lambda)\phi_b$$

By (6.10), the first term on the right maps $\hat{x}^k H^s_{\mathrm{ss},o}(M) \to \hat{x}^l H^{s+1}_{\mathrm{ss},o}(M)$. The third term equals $\sum_b \psi_b\phi_b + Q = \mathrm{Id} + Q$, where Q is a compactly supported pseudodifferential operator of order $-\infty$. Finally, $[L_b, \psi_b]$ is a differential operator with coefficients supported in a set disjoint from $\operatorname{supp}\phi_b$ in $\hat{\mathfrak{a}}$. The result now follows from the previous proposition. □

Theorem 6.3 now follows from Proposition 6.5 and the analytic Fredholm theorem.

When $\theta = 0$, there is an even stronger conclusion:

Theorem 6.6. *The spectrum of Δ_{rad} is the half-line $[|\rho|^2, \infty)$; in other words, there is no point spectrum below the continuous spectrum.*

Proof. Suppose that Δ_{rad} has an eigenvalue $\lambda_1 < |\rho|^2$. Then λ_1 is also an eigenvalue of Δ, the Laplacian on the symmetric space M. By a theorem of Sullivan [22, Theorem 2.1], the existence of a positive solution to $(\Delta - \lambda)u = 0$ is equivalent to $\lambda \leq \inf \mathrm{spec}(\Delta)$, so to prove the theorem we only need provide such a positive solution with $\lambda > \lambda_1$.

To do this, recall the decomposition $G = NAK$, so that $M = G/K$ is identified with NA. We consider the N-invariant solutions of $(\Delta - \lambda)u = 0$. The radial part of Δ with respect to the N-action (i.e. Δ acting on N-invariant functions) has the form $e^{\rho}\Delta_{\mathfrak{a}}e^{-\rho} + |\rho|^2$, see [8, Chapter II, Proposition 3.8]; the discrepancy in signs arises because our Laplacian is the one with positive spectrum. It is thus natural to consider 'plane wave solutions', i.e. those of the form $u(H) = \exp((\rho - \beta)(H))$, $H \in \mathfrak{a}$, where $\beta \in \mathfrak{a}^*_{\mathbb{C}}$ satisfies $-\beta \cdot \beta + |\rho|^2 = \lambda$. When $\lambda \in \mathbb{R}$, $\lambda < |\rho|^2$, then we can take $\beta \in \mathfrak{a}$, and so u is real-valued and everywhere positive. Choosing $\lambda > \lambda_1$, so $\lambda \in (\lambda_1, |\rho|^2)$, completes the proof as noted above. □

We also claim that there are no eigenvalues embedded in the continuous spectrum, i.e. in the ray (λ_0, ∞). This may be proved using N-body techniques, i.e. positive commutator techniques as in [24]. Indeed, [23] proves the corresponding result for first order N-body perturbations of $\Delta_{\mathfrak{a}}$. Unfortunately, while the method requires only trivial modifications, the result does not apply directly due to the apparent singularities at the Weyl chamber walls. Since setting up this approach would require a substantial detour, we postpone this to elsewhere.

It is natural to conjecture that there are no eigenvalues in the resolvent set of $(\Delta_{\mathrm{rad},\theta} - \lambda)^{-1}$ for any θ with $\mathrm{Im}\,\theta < \pi/2$, or in other words, one does not encounter poles of the continued resolvent until one rotates a full angle of π. Furthermore, the poles on the negative real axis should correspond to a spectral problem on the compact dual of M. This can be checked directly when $M = \mathbb{H}^n$. We expect to prove this conjecture using purely analytic arguments, i.e. without resorting to representation theory. The main point is to analyze the limiting operators $\Delta_{\mathrm{rad},\theta}$ when $\mathrm{Im}\,\theta \to \pm\pi/2$; this is nontrivial since the coefficients of this operator develop a number of new singularities in this limit. Roughly, the limiting operators are the radial parts of Laplacians on infinitely many copies of the compact dual, connected by linking 'boundary conditions'. More precisely, $\mathrm{Im}\,\theta \to \pm\pi/2$ is an analytic surgery limit, as described and studied in [11] and [16]: M becomes pinched along the submanifolds where roots α assume values which are non-zero integer multiples of π. This is already seen in the expression (4.1) for the density $\eta\, da$. Such a result would imply the very pleasant consequence that the the domain of analytic continuation has only the single ramification point $|\rho|^2$, and does not inherit the thresholds and eigenvalues from lower rank cases as Regge poles, i.e. new thresholds. Unfortunately but necessarily, the proof would be rather involved, and it has seemed prudent to defer it to another paper.

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