

GLUING AND MODULI FOR NONCOMPACT GEOMETRIC PROBLEMS

RAFE MAZZEO^(†) AND DANIEL POLLACK^(★)

Stanford University (†) and University of Chicago (★)

I. Introduction

In this paper we survey a number of recent results concerning the existence and moduli spaces of solutions of various geometric problems on noncompact manifolds. The three problems which we discuss in detail are:

- I. Complete properly immersed minimal surfaces in \mathbb{R}^3 with finite total curvature.
- II. Complete embedded surfaces of constant mean curvature in \mathbb{R}^3 with finite topology.
- III. Complete conformal metrics of constant positive scalar curvature on $M^n \setminus \Lambda$, where M^n is a compact Riemannian manifold, $n \geq 3$ and $\Lambda \subset M$ is closed.

The existence results we discuss for each of these problems are ones whereby known solutions (sometimes satisfying certain nondegeneracy hypotheses) are glued together to produce new solutions. Although this sort of procedure is quite well-known, there have been some recent advances on which we wish to report here. We also discuss what has been established about the moduli spaces of all solutions to these problems, and report on some work in progress concerning global aspects of these moduli spaces. In the final section we present a new compactness result for the ‘unmarked moduli spaces’ for problem **III**.

Although the analysis underlying each of these problems differs somewhat from one case to the next, there are many common themes. The basic point which makes each of these problems tractable is the fact that the geometry of the ends of the manifolds of interest is well-understood. The ends of a surface Σ in case **I** are either asymptotically planes or catenoids, in case **II** the ends are asymptotically periodic, and in case **III**,

Research supported in part by (†) NSF Young Investigator Award and NSF grant # DMS9303236, and (★) NSF Postdoctoral Research Fellowship.

when Λ is regular, the ends are either asymptotically periodic or asymptotically of ‘edge type’, which means that near infinity they are locally asymptotic to a neighbourhood of infinity in a product of hyperbolic space and a compact manifold. In each of these cases there is a well-developed set of techniques to study the geometric linear elliptic operators on each of these types of manifolds.

The point of view we wish to promulgate here is that gluing methods for constructing new solutions from old ones, and the study of the local structure of the moduli space of all solutions of any one of these problems are most effectively handled by using linear analysis tailored specifically to the geometry at hand. In what follows we survey recent work in these areas, in some of which we have played a role, and attempt to give some idea of the techniques we have found to be successful. One reason for including problem **I** here is the great similarity between the moduli space results obtained by Pérez and Ros [PR], using rather different methods, and our own results in these directions for problems **II** and **III**.

In an attempt to discuss a large number of results by many authors in a brief survey we have undoubtedly given some topics less attention than they would deserve in a more thorough survey of the field. Suffice it to say that our aim is to present those results which we have found most relevant in our attempts to understand and establish new results concerning gluing and moduli for these geometric problems.

2. Gluing

A very rough outline of the gluing method is as follows. If one wishes to construct a new solution of one of the problems **I** – **III**, the first step is to construct an approximate solution. This is built out of pieces of actual solutions already at hand, which are somehow joined together. This approximate solution is usually an actual solution away from the transition regions where the pieces are adjoined. Suppose we are in cases **I** or **II**. Then surfaces near to this approximate solution surface, Σ , may be written as a normal graph off of Σ , i.e. as

$$(2.1) \quad \Sigma_\phi = \{x + \phi(x)\nu(x) : x \in \Sigma\}$$

where $\nu(x)$ is the unit normal to Σ at x and ϕ is small. In case **III** this normal perturbation is replaced by a conformal factor close to 1: metrics near to some g_0 are written as

$$(2.2) \quad g = (1 + v)^{\frac{4}{n-2}} g_0,$$

for some function v which is taken to be small in an appropriate norm. For simplicity we call these perturbations ‘normal perturbations’. We also need to consider nearby surfaces or metrics in a slightly broader sense. Thus we also consider normal perturbations off of not just Σ or g_0 but also surfaces or metrics where the geometry of the ends is slightly changed and allowed to vary slightly within the associated asymptotic family of solutions. For example, for the extrinsic problems **I** or **II** this includes small

independent rotations or translations of each end. We make this more precise below in each case, but the salient fact is that these asymptotic changes form a finite dimensional family. Thus the set of all solutions to the geometric problem near to Σ or g_0 are contained within the set of normal perturbations off of elements of this finite dimensional family of perturbations of the ends.

Amongst all such surfaces or metrics one seeks one which is a solution of the geometric problem. This may be expressed as a second order elliptic partial differential equation in ϕ or v . For **I** and **II** this equation is quasilinear, while for **III** it is only semilinear, but with a critical nonlinearity. The gluing construction is completed if one can find a solution of this nonlinear equation.

There are various difficulties in implementing this method. First, the construction of the approximate solution can be extremely delicate. Second, it can be quite difficult to obtain a solution of the nonlinear equation: frequently (and for geometric reasons) its linearization is not surjective, and in addition one must deal with the fact that the surface or manifold is noncompact. As we discuss below, both of these problems are handled in part by introducing a supplementary finite dimensional space W , the deficiency subspace, which contains the linearizations of the changes in the geometry of the ends mentioned above.

In the remainder of this section we discuss the three cases in turn, surveying the existence results that have been proved using gluing methods and discussing the last case in some detail.

EMBEDDED MINIMAL SURFACES WITH FINITE TOTAL CURVATURE

Amongst the three geometric problems we have mentioned, the one that has received by far the most attention is the first, concerning the existence and nature of complete embedded minimal surfaces in \mathbb{R}^3 . A surface $\Sigma \subset \mathbb{R}^3$ is said to be minimal if its mean curvature H , which at a point $p \in \Sigma$ is the average of the two principal curvatures κ_1 and κ_2 of Σ at p , vanishes. This is equivalent to saying that the surface is a critical point for the area functional under compactly supported variations. The two simplest (complete) examples are the plane and the catenoid. The latter is the surface of revolution obtained by revolving the catenary, $y = \cosh x$ around the x -axis. There is a classical method for generating minimal surfaces using complex function theory, leading to what is known as the Weierstrass representation for minimal surfaces in \mathbb{R}^3 . Given certain holomorphic data, one immediately writes down a minimal immersion. Unfortunately this immersion is rarely a proper embedding, and even if it is, it may well correspond to a surface of infinite topology. For example, there are many explicit singly or doubly periodic minimal surfaces. One of the central questions concerns the existence and nature of complete embedded minimal surfaces for which the integral of the Gauss curvature is finite. Of the classically known examples, only the plane and catenoid have this property. In the early 1980's the first new example was discovered by C. Costa [Co] who exhibited a genus one minimal immersion with one planar and two catenoidal embedded ends. Costa's surfaces was later shown to be embedded by D. Hoffman and W. Meeks III [HM]. This breakthrough led to a huge resurgence

of interest in this area. Since that time many new families of examples of complete embedded minimal surfaces have been discovered. We refer the reader to two very well-written sources, the well-known monograph by R. Osserman [O] and the recent survey paper of D. Hoffman and H. Karcher [HKa], for more information.

An important tool in the study of a complete embedded minimal surface Σ is the Gauss map $\nu : \Sigma \rightarrow \mathbb{S}^2$. The minimality of Σ implies that ν is conformal; this leads to the fundamental fact, first established by Osserman, that Σ is conformally equivalent to the complement of a finite set of points $\{p_1, \dots, p_k\}$ in a *compact* Riemann surface M . In particular, Σ must have finite topology. It is also known that each end of Σ must be asymptotically equivalent to a plane or a catenoid. In order for these ends not to intersect, their axes must be parallel; this is equivalent to the assertion that the limit of ν along each end is one of two fixed antipodal points in \mathbb{S}^2 . We normalize by demanding that these ends are asymptotically horizontal, i.e. that the limiting value of ν at each end is $\pm e_3$.

As noted, the construction of new minimal surfaces has relied primarily on the techniques associated with the Weierstrass representation. A more satisfactory and flexible method would be provided by gluing techniques, and quite recently N. Kapouleas has announced a dramatic new result of this type which we discuss briefly. His result provides a method to ‘desingularize’ a transverse intersection of complete embedded minimal surfaces of finite total curvature. At present he has announced a result when the initial configuration of intersecting surfaces is rotationally invariant, and expects the general case to be completed soon.

The desingularization rests on the existence of a family of simply periodic complete embedded minimal surfaces which desingularize the intersection of two planes meeting at any angle. These surfaces were discovered by Scherk in the 19th century via their Weierstrass representations. They may be visualized by replacing the line of intersection by a periodic string of ‘handles’.

Now, let $\{\Sigma_j\}$ be a finite collection of complete embedded rotationally invariant minimal surfaces of finite total curvature sharing a common axis of rotation. Assume that these surfaces meet transversely and that there are no triple intersections. This set is called the initial configuration. The curves of intersection of the different Σ_j are circles. The intersection $\Sigma_i \cap \Sigma_j$ is desingularized as follows. First an approximate solution is constructed by bending a Scherk surface around this curve and joining its planar ‘wings’ to Σ_i and Σ_j appropriately. This step requires very delicate estimates to show that this can be done in such a way that the error terms incurred in the transition regions are suitably small. These error terms are then dealt with by a careful analysis of the linearized mean curvature, or Jacobi, operator. Kapouleas’ methods for this step are rather different than the ones we outline below. This then allows him to solve the nonlinear problem and find a new complete embedded minimal surface with finite total curvature as a perturbation over the initial configuration.

Recently M. Traizet [Tr] has given a closely related minimal desingularization of the intersection of a finite collection of planes with normals lying in a common two-plane, satisfying certain genericity conditions.

The study of complete surfaces in \mathbb{R}^3 with constant mean curvature (CMC) is almost as old, although it is perhaps not quite as intensively studied. An old theme in the subject was the question of whether it is possible for there to exist a compact embedded or immersed surface without boundary of constant mean curvature other than the sphere. Alexandrov [A] showed that the round sphere is the only such surface which is embedded. For several decades this question remained open, until the early 1980's when Wente [W1] (and subsequently [W2]) constructed constant mean curvature immersions of a torus. This construction has since been reinterpreted and amplified using completely integrable systems; this has led to a classification of all immersed CMC tori, cf. the work of Pinkall and Sterling [PS], Abresch [Ab1] [Ab2], and Bobenko [B].

It turns out that there exist constant mean curvature immersions of compact surfaces without boundary of all genera greater than zero. The existence of these higher genus examples was established in the landmark papers of Kapouleas [K2], [K3]. These papers established the gluing method as a valuable tool in extrinsic geometry. In the first, certain 'balanced' arrangements of spheres and pieces of Delaunay surfaces (described below) are joined together to obtain an immersed approximate solution, and then perturbed to an exact solution. In [K1] he uses a similar method to construct many complete noncompact CMC surfaces, both immersed and embedded. In the compact case only surfaces of genus greater than two may be obtained this way. To remedy this Kapouleas found, in [K3], a more elaborate and general construction whereby Wente tori are fused to each other and to these other pieces. Before describing these results more we need to digress to discuss the noncompact Delaunay surfaces, pieces of which, along with spheres and Wente tori, serve as the other building blocks in Kapouleas' construction.

The simplest noncompact CMC surface is the cylinder. One can ask which other surfaces of revolution have constant mean curvature, and phrased this way it is not difficult to discover a one-parameter family of curves such that the corresponding surfaces of revolution have constant mean curvature. Furthermore these are the only complete CMC surfaces of revolution besides the sphere. These were discovered by C. Delaunay in 1841 [D], and are commonly known as the Delaunay surfaces. The curves producing them are periodic and the embedded surfaces in this family may be parametrized by the minimum distance to the axis of revolution. If we normalize the mean curvature to be 1, then this minimum distance ϵ varies between 0 and 1 and the corresponding surfaces are denoted D_ϵ . Thus D_1 is the unit cylinder and as $\epsilon \rightarrow 0$ the surfaces D_ϵ converge to the singular CMC surface formed by an infinite string of mutually tangent spheres of radius 2 arranged along an axis. By choosing the parametrization somewhat differently we obtain, for $\epsilon < 0$, generating curves which cross the axis of revolution and give rise to Delaunay surfaces which are immersed. There is a very pretty geometric description of this whole family of generating curves: they are the curves traced by one focus of a conic rolling along the axis. For example, the cylinder is obtained as the locus of the center of a rolling circle of radius 1; the embedded generating curves are the loci of rolling ellipses, where, as $\epsilon \rightarrow 0$, the eccentricity of the ellipse tends to

infinity, the singular CMC surface obtained when $\epsilon = 0$ corresponds to ‘rolling’ the infinite eccentricity conic, i.e. the line segment of length 2. Finally, the nonembedded generating curves correspond to rolling hyperbolæ. This is explained in detail in [E].

There are many difficulties in obtaining sufficiently good approximate solutions in Kapouleas’ construction; this step is more subtle than in the intrinsic case **III** discussed below. One starts with a ‘labeled graph’, i.e. a collection of vertices and edges as points and line segments in \mathbb{R}^3 , such that each edge is assigned a real number, which satisfies certain ‘balancing conditions’ involving the geometry of the graph as well as the labeling. One then builds an approximate solution by connecting spheres or pieces of Delaunay surfaces with very small necksize at each vertex or string of vertices, as prescribed by this labeled graph. The edges between vertices correspond to catenoid-shaped necks. Great care is needed in joining these various pieces together to obtain a good approximate solution. The second step is to try to perturb this configuration to an exact solution, and this involves a thorough understanding of the Jacobi operator \mathcal{L} . \mathcal{L} is a second order, self-adjoint elliptic operator which, in cases **I** and **II**, takes the simple form

$$\mathcal{L} = \Delta_\Sigma + |A_\Sigma|^2,$$

where $|A_\Sigma|^2$ is the squared norm of the second fundamental form of $\Sigma \hookrightarrow \mathbb{R}^3$. The main complication in this step is that \mathcal{L} has a number of small or vanishing eigenvalues, caused by the degeneracy, or existence of null modes, of the Jacobi operator on each of the building blocks, i.e. the spheres, Wente tori and Delaunay surfaces. This degeneracy is precisely what separates a hard gluing theorem from some of the softer ones we describe below.

The non-invertibility of \mathcal{L} on the sphere, for example, is a geometric phenomenon which is not difficult to understand. Let Σ_t be a one-parameter family of CMC surfaces deforming $\Sigma_0 = \mathbb{S}^2$ and write Σ_t as a normal graph off of Σ_0 of some function Φ_t . This function solves the CMC equation $N(\Phi_t) = 0$; differentiating this and evaluating at $t = 0$ produces an element of the nullspace of $\mathcal{L} = \Delta_{\mathbb{S}^2} + 2$. The obvious deformations comes from translations in \mathbb{R}^3 , and from these we get a 3-dimensional nullspace of \mathcal{L} on the sphere (or any closed CMC surface in \mathbb{R}^3). These Jacobi fields are the restrictions to the sphere of the linear functions in \mathbb{R}^3 .

To get around the problem of the degeneracy of the Jacobi operator, Kapouleas identifies an explicit approximate nullspace, and by carefully adjusting his approximate solution he generates a ‘substitute kernel’ which in effect allows him to work orthogonally to the approximate nullspace. In the noncompact case this requires infinitely many adjustments. Kapouleas’ work is also briefly discussed in [S4].

It turns out, by a theorem of Korevaar, Kusner, and Solomon [KKS] that any complete embedded CMC surface is in many ways similar to the surfaces that Kapouleas constructs. In particular, each end is strongly asymptotic to an embedded Delaunay surface. The proof of this uses an Alexandrov reflection argument, and depends rather strongly on the embeddedness (or the slightly weaker condition of ‘Alexandrov embeddedness’). In further work of Kusner and Korevaar [KK], it is shown that there exists a graph in \mathbb{R}^3 such that Σ is contained in a neighbourhood of this graph consisting of

the union of tubular neighborhoods of radius 6 about the edges with balls of radius 21 about the vertices.

Very explicit symmetric examples have been constructed by Grosse-Brauckmann [G]. For example, he constructs one-parameter families of surfaces with k ends which have a k -fold dihedral symmetry, and consist of k Delaunay ends joined at a center. On one extreme of this family they meet at a central sphere, while at the other extreme they join at a concave surface which is asymptotically, as the necksizes tend to zero, k -noidal c.f. [G]. Quite recently Grosse-Brauckmann and Kusner [GK] have established new necessary conditions for the existence of certain CMC surfaces with three or four ends possessing some (but not full k -fold dihedral) symmetry. They also have found surfaces of this type experimentally, using a computer; rigorous proofs of their existence would be extremely interesting.

Currently the authors and Kapouleas [KaMP] are developing a gluing procedure to establish an interior bridge principle for nondegenerate CMC surfaces. The method used here combines Kapouleas' ideas for constructing a one-parameter family of approximate solutions with arbitrarily small neck as the bridge and the authors techniques, developed in [MPa1], [MPU2], for perturbing an approximate solution on the connected sum of two nondegenerate solutions to an exact solution. One main ingredient is a uniform analysis of \mathcal{L} as the size of the bridging neck shrinks. Unlike the scalar curvature problem discussed below in detail, the introduction of a small neck automatically leads to the existence of small eigenvalues for \mathcal{L} . This is because the neck is shaped like a catenoid asymptotically, hence the solution of $\mathcal{L}\phi = 0$ on this region may be arbitrarily well approximated by solutions of this equation on the catenoid. But there are solutions of this equation on the catenoid which decay at infinity; they are produced from translations perpendicular to the axis of the catenoid in the manner described above. Thus it seems impossible to develop a gluing method for CMC surfaces which deals only with nondegenerate surfaces, even if the original surfaces to be glued are nondegenerate.

The surfaces produced by this last gluing method are quite different than the ones obtained by Kapouleas earlier. In particular, our method produces a $(3(2k) - 6)$ -parameter family (up to ambient Euclidean motions) of CMC embeddings of the connected sum of k different Delaunay surfaces. This produces an open set in the moduli space of embedded CMC surfaces with $2k$ -ends, even though the resulting surfaces may be degenerate for the reasons indicated above. The necksizes of these surfaces need not be small, although the necks joining the different Delaunay surfaces are quite small. The final perturbation, from the approximate solution to an exact solution, produces a small change in the necksizes on half of each summand, but the necksizes on the other ends are not changed.

Thus for every $k \geq 2$ and a collection of necksizes $(\epsilon_1, \dots, \epsilon_k)$ we prove the existence of a complete embedded surface with constant mean curvature and $2k$ ends such that k of the ends have prescribed necksizes $\epsilon_1, \dots, \epsilon_k$. Included amongst these are the first examples of complete embedded CMC surfaces with cylindrical ends which are not cylinders. Of course, only half the ends of these surfaces are cylindrical; on the

other ends a slight periodicity has been introduced by the final perturbation. A related open question was recently posed to us by R. Schoen: is a complete embedded CMC surface, all of whose ends are asymptotically cylindrical (or equivalently, having finite total absolute Gauss curvature) necessarily a cylinder?

COMPLETE METRICS OF CONSTANT POSITIVE SCALAR CURVATURE

In their study of conformally flat manifolds [SY], Schoen and Yau showed that the developing map of any compact, conformally flat manifold M admitting a compatible metric of positive scalar curvature, is injective as a map from the universal cover of the manifold into the n -sphere. The corresponding holonomy representation is a monomorphism of the fundamental group into the conformal group, $SO(n+1, 1)$. In particular, the universal cover of any such manifold is conformally equivalent to a domain in the sphere. They also showed that the complement of the image of this map in \mathbb{S}^n has Hausdorff dimension less than or equal to $(n-2)/2$. What they actually show is that if $g = u^{4/(n-2)}g_0$ is a complete metric of nonnegative scalar curvature on a domain $\mathbb{S}^n \setminus \Lambda$, where g_0 is the standard metric on the sphere, and if g has bounded Ricci curvature (which is automatically satisfied if g is the lift of a metric on a compact manifold), then $u \in L^{\frac{n+2}{n-2}}(\mathbb{S}^n)$ and u extends to a global weak solution of the equation

$$\Delta_{g_0} u - \frac{n(n-2)}{2}u + \frac{(n-2)}{4(n-1)}R(g)u^{\frac{n+2}{n-2}} = 0.$$

Here $R(g) \geq 0$ is the scalar curvature function of g . This leads to the estimate that $\dim \Lambda \leq (n-2)/2$.

Inspired by this result there has been much subsequent work on case **III**, which is sometimes called the singular Yamabe problem. Here one is given a compact Riemannian manifold (M, g_0) and a closed subset $\Lambda \subset M$, and one seeks a metric g which is conformal to g_0 , complete on $M \setminus \Lambda$ and has constant positive scalar curvature (or CPSC). This is equivalent to asking for a solution of the equation

$$(2.3) \quad \Delta_{g_0} u - \frac{(n-2)}{4(n-1)}R_0 u + \frac{(n-2)}{4(n-1)}R u^{\frac{n+2}{n-2}} = 0$$

on $M \setminus \Lambda$, where u is required to blow up sufficiently strongly on approach to Λ to ensure completeness of g . This last requirement should be regarded as a sort of singular boundary condition (albeit at a ‘boundary’ of high codimension). In this equation R_0 is the scalar curvature function of the background metric g_0 and R is the prescribed constant scalar curvature of the metric $g = u^{4/(n-2)}g_0$.

It is known that in order for there to exist a solution to (2.3) with $R \geq 0$, the background metric must satisfy an extra condition, namely that the first eigenvalue of the conformal Laplacian

$$(2.4) \quad -\Delta + \frac{n-2}{4(n-1)}R(g_0)$$

be nonnegative. This is equivalent to the existence of a metric (without singularities) of nonnegative scalar curvature conformal to g_0 .

There has also been extensive work concerning this problem when the prescribed constant scalar curvature is negative. This has a quite different character than the positive case, and the techniques used to produce solutions are also quite different. For example, one no longer need require any conditions on the conformal class $[g_0]$; also if a (complete) solution with $R < 0$ exists then it is necessary that $\dim \Lambda > (n - 2)/2$. In this case it is possible to use the maximum principle and barrier arguments, and one also finds that solutions are unique. Another difference in this negative case is that solutions no longer extend to global weak solutions on M . A survey of this case is to be found in [Mc], and there is some discussion in [MS], [M2]. Since then there has been some nice recent work by D. Finn [F]. He shows that for any manifold M , if Λ is a finite, possibly intersecting, union of submanifolds with boundary, each with dimension greater than $(n - 2)/2$, then there exists a complete metric of constant negative scalar curvature on $M \setminus \Lambda$. Recently Finn has also obtained results concerning existence and nonexistence when Λ is a more general stratified set. We remark that the issue in this case is always about the completeness of the metric g , for it is relatively straightforward to produce solutions to (2.3) with $R < 0$ by barrier arguments.

When $R = 0$, (2.3) reduces to a linear equation, and the techniques of [M1] may be used directly, at least when Λ is a (possibly infinite and intersecting) union of submanifolds without boundary. It is necessary that the dimension of each component of Λ is less than or equal to $(n - 2)/2$.

Entirely different ideas must be used to produce complete metrics of constant positive scalar curvature $M \setminus \Lambda$. In this context the use of gluing techniques to produce new solutions began with Schoen's deep construction [S] of complete conformally flat metrics on $\mathbb{S}^n \setminus \Lambda$, where Λ is either an arbitrary finite set, of cardinality greater than one, or certain nonrectifiable 'limit sets'. There are many similarities between this and Kapouleas' first construction of CMC surfaces [K1], which was developed around the same time, although in many technical aspects the two cases are quite different. The building blocks here are spheres of radius one, and an approximate solution is formed by joining together infinitely many of these in a certain balanced configuration, connecting adjacent spheres by very small necks. Just as in the CMC case, one must compensate for the fact that these building blocks are degenerate in the sense that the linearization of (2.3), \mathcal{L} , has nullspace on each of them. These nullspaces together span an infinite dimensional space X such that the restriction of \mathcal{L} to X has arbitrarily small norm relative to the neck size if all necks in the approximate solution are sufficiently small. When R is normalized to equal $n(n - 1)$, this operator takes the form

$$(2.5) \quad \mathcal{L} = \Delta + n.$$

To perturb these approximate solutions to an exact solution, Schoen solves the non-linear problem orthogonally to X and then shows that if the initial configuration is chosen correctly this is actually a solution to the full problem. More detailed exegeses of this construction are given in [MPU1, §7] and [P1]. A similar approach was used

by the second author in [P1] to demonstrate nonuniqueness and construct high energy solutions for the Yamabe problem on compact manifolds.

There are simple examples of CPSC metrics on $\mathbb{S}^n \setminus \mathbb{S}^k$: the standard metric is conformally equivalent, by stereographic projection from some point on \mathbb{S}^k , to the flat metric on $\mathbb{R}^n \setminus \mathbb{R}^k$; writing this in cylindrical coordinates (r, θ, y) around \mathbb{R}^k as $dr^2 + r^2 d\theta^2 + dy^2$, we conformally change again by dividing by r^2 to obtain the product metric $r^{-2}(dr^2 + dy^2) + d\theta^2$ on $\mathbb{H}^{k+1} \times \mathbb{S}^{n-k-1}$. This is a product of constant curvature spaces, and has constant scalar curvature $R = (n-2)(n-2k-2)$ which is positive if and only if $k < (n-2)/2$.

When $k = 0$ the CPSC metric we obtain this way is the cylinder. Just as for CMC surfaces discussed earlier, the cylinder lies in a one-parameter family of spherically symmetric metrics, all conformally related to one another, which are periodic and degenerate to an infinite set of spheres of radius 1, mutually tangent and lying along a fixed axis. In analogy to the CMC case we call these Delaunay metrics D_ϵ , but they correspond to solutions of an ODE which was first studied by Fowler [Fo1], [Fo2]. This ODE is rather simple: using the metric $dt^2 + d\theta^2$ on the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$, then a conformally related metric $u^{4/(n-2)}(dt^2 + d\theta^2)$ is spherically symmetric and has CPSC provided u depends only on t and

$$(2.6) \quad \partial_t^2 u - \frac{(n-2)^2}{4} u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0.$$

This equation may be analyzed by writing it as a first order system for u and $\dot{u} = \partial_t u$. The Hamiltonian energy

$$(2.7) \quad H(u, \dot{u}) = \frac{\dot{u}^2}{2} - \frac{(n-2)^2}{8} u^2 + \frac{(n-2)^2}{8} u^{\frac{2n}{n-2}}$$

is constant along orbits of this system. The solutions lying within the set where $H = 0$ are $u \equiv 0$ and $u \equiv \cosh^{(2-n)/2} t$, the latter of which corresponds to the metric $(\cosh t)^{-2} (dt^2 + d\theta^2)$, which is the incomplete spherical metric on $\mathbb{S}^n \setminus \mathbb{S}^0$. All solutions of (2.6) which remain positive for all t remain within the bounded set $\{H < 0\}$, and correspond to periodic orbits, which are the Delaunay-Fowler solutions. The stationary point in this set, where u is constant, corresponds to the product metric on the cylinder, rescaled to have scalar curvature $n(n-1)$.

One reason for the importance of the Delaunay metrics is the result, analogous to the result of [KKS] for CMC surfaces, that if g is a conformally flat metric on a ball with an isolated singularity at the origin, then g is asymptotic at its singular point to a unique Delaunay solution D_ϵ . This was proved initially by Caffarelli-Gidas-Spruck [CGS], and later reproved, with a better estimate of convergence, by Aviles, Korevaar and Schoen [AKS]. Another proof of the Caffarelli-Gidas-Spruck estimate has recently been given by C.C. Chen and C.S. Lin [CL2]. The improvement of this estimate to the one obtained by Aviles, Korevaar and Schoen is exactly analogous to the argument given in [KKS] for case **II**. It is unknown whether this result holds for nonconformally

flat metrics, although it is easy to produce, by perturbation methods, nonconformally flat metrics on a punctured ball with Delaunay asymptotics at the origin.

By this asymptotics result we see in particular, that any of the solutions produced by Schoen [S1] on $\mathbb{S}^n \setminus \Lambda$, where Λ is a finite set, are metrics with asymptotically Delaunay ends. The asymptotic Delaunay parameters which arise in Schoen's construction are all very close to 0.

The local asymptotics near a higher dimensional singularity may be much more complicated. It turns out that the product solutions on $\mathbb{H}^{k+1} \times \mathbb{S}^{n-k-1}$ may be extended to solutions periodic with respect to certain discrete groups of hyperbolic motions. This is accomplished by regarding this metric as the universal cover of some compact manifold $X_\Gamma = (\mathbb{H}^{k+1}/\Gamma) \times \mathbb{S}^{n-k-1}$. This manifold has CPSC, but it is possible to show [M2] that for certain choices of Γ this product hyperbolic-spherical metric does not minimize the Yamabe energy, hence there is another CPSC metric which does minimize this energy; its lift to $\mathbb{S}^n \setminus \mathbb{S}^k$ is periodic with respect to the Γ action on \mathbb{H}^{k+1} , and it may be shown that it is constant along the \mathbb{S}^{n-k-1} factors.

It is unknown whether these periodic solutions are anomalous, or whether they exist on more general sets $M \setminus \Lambda$ where Λ is a k -dimensional submanifold. In general it is possible to use perturbation methods to include a 'nondegenerate', 'admissible' solution metric on $M \setminus \Lambda$ into an infinite dimensional family (in fact a Banach manifold) of solutions with the same singular set, where Λ is a nonintersecting union of submanifolds of dimensions between 0 and $(n-2)/2$, at least one of which has positive dimension. Here nondegenerate means that the linearized scalar curvature operator $\Delta_g + n$ has no L^2 nullspace. Admissible means that the conformal factor u (relative to the background metric g_0 which extends smoothly across Λ) has the asymptotic form $u \sim Ar^{(2-n)/2}(1 + o(1))$, where A is a purely dimensional constant. These perturbation methods were first developed for the product metrics on $\mathbb{S}^n \setminus \mathbb{S}^k$ in [MS], but they may also be applied to the solution metrics constructed in [MPa1]. The Γ -periodic solutions discussed above are not admissible, and it would be extremely interesting, and probably quite difficult, to determine if they are nondegenerate, hence deformable to a family of asymptotically periodic metrics.

We should mention that if a solution metric g is admissible, then it is proved in [M2] that u admits a partial expansion near any component of Λ with positive dimension k of the form

$$u \sim Ar^{(2-n)/2} \left(1 + r^{k/2+i\mu} v_+ + r^{k/2-i\mu} v_- + o(r^{k/2}) \right).$$

The leading asymptotics v_+ , v_- are functionally related via a sort of nonlinear scattering matrix; when they are smooth, there is a full asymptotic expansion, but in general they are only distributional (of negative order!). Another interesting problem would be to show that there is a well-posed Dirichlet problem, i.e. that there is a solution with v_+ , say, prescribed arbitrarily. It is possible to show that the set of pairs (v_+, v_-) form a Lagrangian submanifold of an appropriate infinite dimensional function space, so what is required for the Dirichlet problem is to determine whether this Lagrangian projects bijectively onto the first 'coordinate'.

When dealing with solutions without such asymptotics one may seek to determine general upper and lower bounds for solutions, in terms of some negative power of the distance to the singular set. There is a general upper bound for solutions on $\mathbb{S}^n \setminus \Lambda$ of the form $u \leq Cd^{(2-n)/2}$, where C is independent of Λ and u , and d is the distance function to Λ [P2]. This does not require any regularity of Λ , but it does use the global completeness of the metric in a crucial way. Recently C.C. Chen and C.S. Lin [CL1] obtained an upper bound of this form for conformally flat solutions defined only within a ball. Lower bounds are much more subtle, and known in only a few cases. For example, for conformally flat solutions with isolated singularities, lower bounds (depending on the solution) are known. Such solutions are asymptotically Delaunay and there are lower bounds *depending on the necksize* for Delaunay solutions. Actually, determining some lower bound for a solution with isolated singularity is an important step in showing that it is asymptotically Delaunay. For singular sets Λ of positive dimension, the results of [MPa1] show that there cannot be lower bounds independent of the solution.

Recently several existence results have been proved for this problem using gluing methods. Initially, Pacard [Pa] showed that there are solutions of this problem on $\mathbb{S}^n \setminus \Lambda$ where Λ is a finite disjoint union of submanifolds of dimension $(n-2)/2$. The paradigm is the usual one: first he constructs a one-parameter family of approximate solutions which become more and more concentrated around Λ and then shows that when these approximate solutions are sufficiently concentrated, they may be perturbed to exact solutions. The crucial observation here is that approximate solutions may be obtained by pasting radial singular solutions of the equation $\Delta u + u^{(n+2)/(n-2)} = 0$ on \mathbb{R}^N , $N = n - (n-2)/2 = (n+2)/2$, into each fibre of the normal bundle of Λ . The exponent $(n+2)/(n-2)$ is subcritical in \mathbb{R}^N , and the equation has solutions which have damped oscillations as $r \rightarrow \infty$ (as opposed to the solutions of the ‘frictionless’ critical exponent equation (2.3) whose oscillations are undamped, i.e. periodic) and are concentrated more and more at the origin. The error term incurred by pasting these into the fibres of the normal bundle of Λ goes to zero as they become more concentrated. The importance of the dimension $(n-2)/2$ is that the equation to be solved is of the form $\Delta_{\mathbb{S}^n} u + Vu = 0$, where the singularity of V is relatively weak, so that classical methods may be used to obtain a solution. This method was extended by Rebaï [Reb] to consider the case when the dimensions of the components of Λ are allowed to be slightly less than $(n-2)/2$. These methods are limited by the fact that only when this dimension is greater than some $k_0 < (n-2)/2$ do the approximate solutions have finite instability. More specifically, the linearized scalar curvature operator at one of these solutions has only finitely many eigenvalues less than zero. By contrast, if the dimension is less than k_0 then the instability becomes infinite; indeed, in these latter cases 0 lies in the middle of a band of continuous spectrum. Nevertheless, replacing the classical methods in this argument by the pseudodifferential edge theory developed in [M1], the first author and Pacard were able to accomplish the second step for any positive k less than $(n-2)/2$, perturbing the approximate solution to an exact solution; in fact, this argument works for any background manifold M of nonnegative scalar curvature.

This argument has many delicate points not encountered in the previous cases, since a subtle harmonic analysis of the subcritical equation in \mathbb{R}^N for radial solutions singular at the origin is now required.

The result of [MPa1] may be regarded as saying that the radial singular solutions on \mathbb{R}^N , transplanted to an ϵ -neighbourhood of the singular set Λ , may be glued to the zero ‘solution’ (which is of course not even a metric) on the complement of this tubular neighbourhood. The crucial point is that both of these solutions which are being glued are nondegenerate in the sense that the nullspaces of their Jacobi operators contain no elements with the appropriate growth restrictions. This argument was recently reinterpreted to prove a rather general result [MPU2] that from any two nondegenerate CPSC metrics, (M_1, g_1) and (M_2, g_2) , which are either complete or where one or both of the M_i have boundary, one may construct a (complete) CPSC metric on the connected sum $M_1 \# M_2$. We wish to discuss this result a bit further, and in particular to give the correct general notion of nondegeneracy.

CONNECTED SUMS AND NONDEGENERACY

The Jacobi operator \mathcal{L} (with Dirichlet boundary conditions if $\partial M \neq \emptyset$) of a complete CPSC metric (M, g) is self-adjoint on L^2 . Assume that there is some weight function α on M such that for the corresponding weighted Sobolev spaces

$$H_\gamma^s = \{\phi = \alpha^\gamma \tilde{\phi} : \tilde{\phi} \in H^s\}$$

one has that

$$\mathcal{L} : H_{-\delta}^{s+2} \longrightarrow H_{-\delta}^s$$

is injective with closed range for $\delta > 0$. Our actual hypothesis is that for all $s \in \mathbb{R}$ there is an estimate

$$(2.8) \quad \|\phi\|_{s+2, -\delta} \leq C \|\mathcal{L}\phi\|_{s, -\delta}$$

for all $\phi \in H_{-\delta}^{s+2}(M)$. Note that (2.8) implies both the closed range and the injectivity. The dual of $H_{-\delta}^s$ is H_δ^{-s} , hence (2.8) also implies that \mathcal{L} is surjective (and in particular has closed range) on H_δ^s for all $s \in \mathbb{R}$. Thus if $f \in H_\delta^s$ then there is a solution $v \in H_\delta^{s+2}$ to $\mathcal{L}v = f$. To complete the definition of nondegeneracy we also assume that either the nonlinear scalar curvature operator satisfies

$$(2.9) \quad N : H_\delta^{s+2} \longrightarrow H_\delta^s$$

or else that there exists a ‘deficiency space’ W and a natural extension of N so that

$$(2.9') \quad N : H_\delta^{s+2} \oplus W \longrightarrow H_\delta^s$$

is a well defined, analytic map with surjective linearization

$$(2.10) \quad \mathcal{L} : H_\delta^{s+2} \oplus W \longrightarrow H_\delta^s.$$

These nondegeneracy hypotheses (2.8) - (2.10) on the summands M_1 and M_2 are sufficient to ensure that the construction of a metric of CPSC on the connected sum $M_1 \# M_2$ may be carried out [MPU2]. For the specific types of metrics we have discussed above, we remark that the hypotheses (2.9') and (2.10) are unnecessary for the solution metrics on $M \setminus \Lambda$ from [MPa1] when all components of Λ have positive dimension. It is unknown whether the metrics produced by Schoen [S1] on the complement of a finite set in \mathbb{S}^n satisfy (2.8) - (2.10). However, the Delaunay solutions do satisfy these conditions. To see this, we first define the deficiency space W for a Delaunay solution D_ϵ . The elements of W are formed from the localizations to each end of D_ϵ of the temperate Jacobi fields. The Jacobi operator on D_ϵ takes the form, in cylindrical coordinates

$$\mathcal{L}_\epsilon = \Delta_{g_\epsilon} + n = u_\epsilon^{-\frac{2n}{n-2}} \partial_t (u_\epsilon^2 \partial t) + u_\epsilon^{-\frac{4}{n-2}} \Delta_\theta + n.$$

Here $g_\epsilon = u_\epsilon^{4/(n-2)} (dt^2 + d\theta^2)$. Localizing to each eigenspace of Δ_θ leads to an operator with two solutions Φ_k^\pm , indexed by the eigenvalues λ_k of Δ_θ , one of which (when $k \neq 0$) decays exponentially as $t \rightarrow -\infty$ and grows exponentially as $t \rightarrow \infty$, and the other of which grows and decays, at the same rate, in the opposite directions. When $k = 0$ the two solutions no longer decay or grow exponentially. Indeed,

$$(2.11) \quad \Phi_0^+ = \partial_t u_\epsilon(t), \quad \Phi_0^- = \partial_\epsilon u_\epsilon(t).$$

Thus these Jacobi fields are integrable in the sense that they correspond to one-parameter families of CPSC metrics on the cylinder: Φ_0^+ corresponds to the family of 'translations' of g_ϵ , while Φ_0^- corresponds to the family where the Delaunay parameter ϵ varies. The only other Jacobi fields which are integrable in this sense are the ones corresponding to the first nonzero eigenvalue on \mathbb{S}_θ^{n-1} .

For a solution g on $\mathbb{S}^n \setminus \Lambda$, with $\Lambda = \{p_1, \dots, p_k\}$, a neighbourhood of each singular point p_j is asymptotic to a Delaunay metric g_{ϵ_j} . Thus the truncated Jacobi fields $\chi \Phi_0^\pm$, for χ a cutoff function, on D_{ϵ_j} may be transplanted to a neighbourhood of p_j . The deficiency space W is the $2k$ -dimensional linear span of these transplanted truncations at all singular points. With this definition it is proved in [MPU1] that condition (2.10) holds if the Jacobi operator has a trivial L^2 nullspace. To verify (2.9') we appeal to the fact that the elements of W are asymptotically integrable, i.e. that up to an error term in $H_{-\delta}^s$ they correspond to one-parameter families of CPSC metrics on small neighbourhoods of the p_j . Thus, for a small element $w \in W$ we may form a metric $g_\epsilon(w)$ which has CPSC on the complement of a neighbourhood of the punctures, and has asymptotically CPSC at each p_j , but the asymptotic Delaunay model at p_j has been translated slightly and its Delaunay parameter slightly changed according to the coefficients of Φ_0^\pm at p_j in w . With this definition (2.9') now holds.

The general gluing result of [MPU2] states that if (M_1, g_1) and (M_2, g_2) satisfy the hypotheses (2.8) - (2.10), and if $M_\eta \equiv M_1 \#_\eta M_2$ is the connected sum, formed by removing small discs of radius $\sqrt{\eta}$ around two specified points $q_j \in M_j$ and identifying the boundaries, then there is a small perturbation of the resulting approximate

solution metric g_η which has CPSC. Furthermore, the resulting CPSC metric is also nondegenerate.

The main part of the proof proceeds in two steps. Denoting by \mathcal{L}_η the Jacobi operator on the approximate solution g_η , we first show that \mathcal{L}_η is injective on $H_{-\delta}^s(M_\eta)$ for η sufficiently small. This means that there is a right inverse (which is not unique) for \mathcal{L}_η as an operator on H_δ^s . The second step is to show that the norm of an appropriate choice of right inverse G_η does not blow up as $\eta \rightarrow 0$. The appropriate choice of right inverse is the one with range lying in the orthogonal complement of the kernel of \mathcal{L}_η on H_δ^s . This orthogonal complement may be identified with the range of the adjoint of \mathcal{L}_η . The way we prove both of these steps is to assume they are not true for arbitrarily small η and to arrive at a contradiction. Thus, for step one, we assume that there is a sequence η_j tending to zero and an element $\phi_j \in H_{-\delta}^s(M_{\eta_j})$ such that $\mathcal{L}_{\eta_j}\phi_j = 0$. If this were true then we could obtain, after some normalizations, a nonzero solution $\phi \in H_{-\delta}^s$ on either M_1 or M_2 satisfying $\mathcal{L}\phi = 0$, and this contradicts the nondegeneracy hypothesis (2.8) on either of these summands. For step two, we assume that there are functions $f_j \in H_\delta^s$, $\phi_j \in H_\delta^{s+2}$ and $v_j \in H_\delta^{s+4}$ such that

$$\mathcal{L}_{\eta_j}\phi_j = f_j, \quad \mathcal{L}_{\eta_j}^*v_j = \phi_j$$

where $\mathcal{L}_{\eta_j}^*$ is the adjoint of \mathcal{L}_{η_j} , with $\|f_j\| \rightarrow 0$ and $\|\phi_j\| \equiv 1$. By again taking limits in a judicious manner, we obtain functions $v \in H_\delta^{s+4}$, $\phi \in H_\delta^{s+2}$ satisfying $\mathcal{L}\phi = 0$, $\mathcal{L}^*v = \phi$ on either M_1 or M_2 , which is again a contradiction. A small additional argument is needed to show that the map (2.10) is surjective on M_η for η sufficiently small and that it too is ‘uniformly surjective.’ The remainder of the theorem, the actual perturbation of the approximate solution to an exact solution, is now straightforward using (2.10).

The point we wish to emphasize in this argument is that no explicit estimates beyond (2.8) are required. Although we do not obtain explicit bounds for the right inverse G_η , these are not really required. This general paradigm, including the indirect arguments we have sketched, represents a substantial simplification of the general gluing procedure, at least when the summands are nondegenerate.

It is not clear whether any of the solutions constructed by Schoen on $\mathbb{S}^n \setminus \Lambda$ are nondegenerate in this sense. More specifically, we have indicated that (2.9’) and (2.10) hold for these CPSC metrics provided (2.8) is valid. It is shown in [MPU1] that the nullspace of \mathcal{L} on $H_{-\delta}^s(\mathbb{S}^n \setminus \Lambda)$, where Λ is a finite set, is always finite dimensional. Because of the parametrices constructed for \mathcal{L} in that paper, to verify (2.8) it is sufficient to know that this nullspace is actually trivial. The only cases where this was known explicitly were the Delaunay solutions themselves. Thus the soft gluing theorem of [MPU2] may be applied to show that k different Delaunay metrics $D_{\epsilon_1}, \dots, D_{\epsilon_k}$ may be glued together to produce a CPSC metric on the complement of $2k$ points in \mathbb{S}^n . Moreover, by being more careful about the definition of the deficiency space W in the construction, we show there that it is possible to produce this glued solution in such a way that the Delaunay parameters ϵ_j are unchanged on one end of each D_{ϵ_j} . This produces the first known examples of CPSC metrics other than the cylinder with

at least some cylindrical ends. The analogue of Schoen’s problem mentioned earlier for CMC surfaces, about whether a singular Yamabe metric on $\mathbb{S}^n \setminus \Lambda$ with all ends asymptotically cylindrical must in fact be a cylinder, is also open.

This general paradigm for gluing can be applied in some degenerate problems too. Recently the first author and Pacard [MPa2] have been able to reprove Schoen’s theorem concerning solutions with isolated singularities and extend it to general manifolds with nonnegative scalar curvature. Uhlenbeck [U] has announced a proof of the existence of solutions on $M \setminus \{p\}$ for any nonnegative scalar curvature manifold M which is not conformally equivalent to the standard sphere (where no such solution exists). The point of both these papers is that while Schoen needs to use infinitely many balancing conditions in his construction, essentially one for each small neck connecting the infinitely many spheres, really only finitely many balancing conditions are needed. These solutions are constructed by gluing half-Delaunay metrics onto the zero solution of M . The exact solution will then be very small, but positive, away from the points $\{p_1, \dots, p_k\}$ and asymptotically Delaunay near each p_j . The reason the degeneracy can be handled is that the elements of the relevant deficiency space W (which is slightly larger than the one we discussed earlier) all correspond to specific geometric changes of asymptotic geometry.

3. Moduli spaces

Once the existence of solutions for any of the problems has been obtained, it is natural to study the moduli space, or the set of all solutions. Unlike in the previous section we now discuss all three cases together. The moduli space for solutions of problem **I** was recently analyzed by Ros [Ros] and Pérez and Ros [PR], while for problems **II** and **III** it was studied in [KMP], [MPU1] and [MPU2]. The methods used by Pérez and Ros were slightly different than those used in the other three papers, but it is possible to extend these other methods to case **I** as well.

In what follows we let $\mathcal{M}_{\text{Min},g,k}$, $\mathcal{M}_{\text{CMC},k}$ and \mathcal{M}_Λ denote the moduli spaces of complete, properly immersed minimal surfaces of finite total curvature with k horizontal ends and genus g , of complete embedded constant mean curvature surfaces with k ends, and of complete metrics of constant positive scalar curvature on $M \setminus \Lambda$ where $\Lambda = \{p_1, \dots, p_k\}$ respectively. In cases **I** and **II** we consider two surfaces the same if they differ by a rigid motion (thus the dimension of $\mathcal{M}_{\text{Min},g,k}$ in (3.1) below is slightly different than in [PR]). When we discuss results that apply to any of the three cases, we often write simply \mathcal{M}_k for the relevant moduli space. (Later we also discuss the ‘unmarked moduli space’ for case **III**, which will be denoted \mathbb{M}_k .) We do not discuss the moduli space of solutions for problem **III** when Λ has positive dimension. As we have already indicated, in this case there is an infinite dimensional space of solutions to the problem. There is hope that this infinite dimensional moduli space may be parametrized using an asymptotic Dirichlet problem, but this is as yet unknown.

GENERAL FEATURES OF THE MODULI SPACES

The basic results proved in these papers are that in each of these three cases \mathcal{M}_k

is locally a real analytic set, and when $\Sigma \in \mathcal{M}_k$ is a nondegenerate element, then a neighbourhood \mathcal{U} of Σ is a real analytic manifold. The formal dimensions, which are the actual dimensions of these neighbourhoods \mathcal{U} around nondegenerate points, are as follows:

$$(3.1) \quad \dim \mathcal{M}_{\text{Min},g,k} = k, \quad \dim \mathcal{M}_{\text{CMC},k} = 3k - 6, \quad \dim \mathcal{M}_\Lambda = k.$$

To say that \mathcal{M}_k is a locally real analytic set of dimension d means that near any $\Sigma \in \mathcal{M}_k$ there is a neighbourhood \mathcal{V} in the space of all nearby surfaces or metrics (which are not necessarily solutions) and a real analytic diffeomorphism $\Psi : \mathcal{V} \rightarrow \tilde{\mathcal{V}}$ to an open set in some finite dimensional space \mathbb{R}^m , such that $\Psi(\mathcal{V} \cap \mathcal{M}_k)$ is the zero set of a real analytic function $F : \tilde{\mathcal{V}} \rightarrow \mathbb{R}^\ell$, where $m - \ell = d$. Thus if the differential of F at $\Psi(\Sigma)$ is surjective, then $\Psi(\mathcal{V} \cap \mathcal{M}_k)$ is a smooth real analytic manifold of dimension d .

The main point that needs to be understood in each case is the possible asymptotic behaviour of solutions. For case **I** this is well-known, cf. [HKa]; the ends of any complete minimal surface in \mathbb{R}^3 are either asymptotically planar or catenoidal. Moreover, under the requirement that the surface is embedded, these ends must be parallel. By only considering horizontal ends one is simply rotating the surface so that the vertical axis is parallel to the axes of these asymptotic planes or catenoids. The analogous results for complete embedded CMC surfaces, and for singular Yamabe metrics on $\mathbb{S}^n \setminus \Lambda$, namely that each end is asymptotically Delaunay, are provided by [KKS] and [CGS], [AKS], [CL2] respectively.

Because of these results, it is possible to parametrize all nearby candidate surfaces or metrics, and to identify a real analytic mapping of which the solutions, i.e. the elements of \mathcal{M}_k , constitute the zero set. More explicitly, we use the deficiency space W introduced in the last subsection for problem **III**, and its analogues for the other cases. The deficiency space in each case is the set of Jacobi fields that correspond to one-parameter families of solutions of the models for each end, suitably truncated and transplanted to the appropriate end. In each of these problems, these one-parameter families arise via rigid motions and changes of the ‘shape’ parameter.

For problem **I** the two models are the plane and the catenoid. The plane should be thought of as a limiting version of the catenoid, as the parameter measuring the logarithmic growth of the catenoid tends to one of its end-values. All catenoids are the same up to translation, rotation and dilation, so this parameter can be identified with the diameter of the neck of the catenoid. We shall call it the logarithmic growth parameter, with the understanding that it takes the value zero for the plane. Since we are restricting attention to surfaces with horizontal ends, the relevant motions of each end are translations, either horizontal or vertical, and change in the logarithmic growth parameter. By comparing each end to a fixed horizontal plane in \mathbb{R}^3 , we may associate two natural parameters to each end: the height (e.g. from the central waste of the catenoid to the fixed plane in the second case) and the logarithmic growth. We may define the deficiency space W in this case as consisting of truncations of

the asymptotic Jacobi fields corresponding to these changes of each end; it is $2k$ -dimensional. The Jacobi field corresponding to changing the height is bounded, and in fact tends to the change of height, while the Jacobi field corresponding to changing the logarithmic growth is linearly growing. By contrast, the asymptotic Jacobi fields corresponding to horizontal translations are decaying and are not included in W . Let \mathcal{K}_0 be the space of all Jacobi fields which decay along each end. This may be rather large, for example if there are deformations of the surface where some ends translate horizontally relative to others. (It is unknown if this phenomenon can occur.) In any case, $\dim \mathcal{K}_0 \geq 3$, because there are always decaying Jacobi fields corresponding to the two horizontal translations and to rotation about the vertical axis. Notice that we do not consider the two other asymptotic Jacobi fields on each end, corresponding to rotations not preserving the vertical direction, because we are restricting attention to minimal surfaces with horizontal ends.

Pérez and Ros show, using what amounts to the relative index computation sketched below, that the number ℓ of polynomially bounded Jacobi fields on Σ is greater than or equal to $k + 3$; in fact, $\ell - (k + 3) = \dim \mathcal{K}_0 - 3$. Σ is called nondegenerate when $\ell = k + 3$. Hence, when Σ is nondegenerate, all the decaying Jacobi fields correspond to global rigid motions of Σ . Even though the Jacobi operator as in (3.2) below is not surjective when Σ is nondegenerate, the obstruction to surjectivity then consists of the decaying Jacobi fields, which are ‘geometrically integrable’. It is not hard to add in these global rigid motions to make the Jacobi operator surjective, and once this is done, Pérez and Ros use the implicit function theorem to show that $\mathcal{M}_{\text{Min},g,k}$ is $k + 3$ -dimensional (or just k -dimensional if surfaces differing by rigid motions are identified) in a neighbourhood of a nondegenerate surface Σ .

For problem **II** there are no global decaying Jacobi fields coming from rigid motions, but there are Jacobi fields, both globally on Σ and for each asymptotic model, arising from the change of Delaunay parameter (which is the analogue of scaling the catenoid) and from all translations and those rotations orthogonal to the asymptotic axes. Thus the deficiency space W here is a $6k$ -dimensional space; for each end there is one Delaunay parameter, three translation parameters and two rotation parameters. Now nondegeneracy is the condition that there are no L^2 Jacobi fields at all. Analogous to the dimension count of [PR], in this case we prove that the dimension ℓ of polynomially bounded Jacobi fields is greater than or equal to $3k$. In the nondegenerate case, this produces a $3k$ -dimensional moduli space, although once again identifying surfaces up to rigid motions we obtain the dimension $3k - 6$ as in (3.1).

Finally, case **III** appears slightly different since it is intrinsic. The only polynomially bounded Jacobi fields for the asymptotic models of each end, i.e. the Delaunay metrics, are the Jacobi fields Φ_0^\pm we have already discussed; these correspond to changing the Delaunay parameter or translating along the axis of the Delaunay solution. This translation is really with respect to the conformal dilation fixing the two singular points of the Delaunay metric. Other possible conformal translations or rotations, because they move the singular points, correspond to exponentially growing Jacobi fields. This is in contrast to the other cases above, where these geometric Jacobi fields exhibit only

linear growth. Thus in this case the deficiency space W is a $2k$ -dimensional space, corresponding to the Delaunay and translation parameters. Below we shall discuss the ‘unmarked moduli space’ for problem **III**, where the singular points are no longer required to remain fixed. Then one must consider all Jacobi fields arising from the conformal action.

To reiterate, the similarities in these three cases is that each end has an asymptotic model, and to each model is associated a set of Jacobi fields which arise geometrically. One now builds a finite dimensional space W of asymptotic Jacobi fields on Σ , using a partition of unity to transplant these model Jacobi fields to each end. W is called the deficiency subspace. The first main result is that if Σ is nondegenerate then

$$(3.2) \quad \mathcal{L} : H_{-\delta}^{s+2} \oplus W \longrightarrow H_{-\delta}^s$$

is surjective (in case **I** we modify this as discussed above). The way to prove this is to first observe that by nondegeneracy and duality, \mathcal{L} is surjective as a map from $H_{\delta}^{s+2} \rightarrow H_{\delta}^s$. Hence if $f \in H_{-\delta}^s$, then there is an element $u \in H_{\delta}^{s+2}$ such that $\mathcal{L}u = f$. The second step is to prove a regularity result for this linear equation, namely that because f decays, u can be decomposed as a sum of functions $u = v + w$, where $v \in H_{-\delta}^{s+2}$ and $w \in W$. This is proved in [MPU1] for case **III**, and the same proof works in case **II**. These results in case **I** are older, and can be found, for example, in [Mel]. The Fredholm theory for Laplacians on asymptotically Euclidean or catenoidal manifolds, or for elliptic operators with the same types of asymptotic behaviour, dates back to work of Lockhart and McOwen, and Melrose; the decomposition lemma in this context is found explicitly in [M1].

Now, as described earlier for case **III**, the nonlinear (mean or scalar curvature) operator N can be defined on $H_{-\delta}^{s+2} \oplus W$ as follows. The elements $w \in W$ of small norm correspond to nearby surfaces or metrics, $\Sigma(w)$ or $g(w)$, which are altered on each end according to the components in w of the truncated Jacobi fields. Thus we obtain these surfaces or metrics by rotating, translating or changing the relevant parameter (size of the catenoid or Delaunay parameter) for each end, then reattaching this slightly altered end to the main body of the surface in some fixed manner, using cutoff functions. The set of all surfaces or metrics which are nearby to Σ or g may be written as normal perturbations by some decaying function $\phi \in H_{-\delta}^{s+2}$ off $\Sigma(w)$ or $g(w)$. Thus we may define

$$(3.3) \quad N(\phi, w) = N_{\Sigma(w)}(\phi), \quad \text{or} \quad N(\phi, w) = N_{g(w)}(\phi).$$

The proof that \mathcal{M}_k is a real analytic manifold of dimension d (one of the three values above) in a neighbourhood of a nondegenerate point is now immediate. The nonlinear operator in (3.3) has surjective linearization (3.2), and the set of solutions of $N(\phi, w) = 0$ in a small neighbourhood of zero is a finite dimensional real analytic manifold, by the implicit function theorem. The dimension of this manifold is the dimension of the nullspace of the map (3.2). To determine this we proceed as follows. Because \mathcal{L} is surjective as a map on H_{δ}^{s+2} the cokernel is trivial, hence the dimension

of this nullspace is the same as the index of this map, i.e. the dimension of the space of solutions of $\mathcal{L}\phi = 0$ in H_δ^s minus the dimension of the cokernel. Even when Σ or g is not nondegenerate, this index is equal to the dimension of the ‘bounded nullspace’ \mathcal{B} , which we take as the orthogonal complement of the nullspace of \mathcal{L} in $H_{-\delta}^s$ in the nullspace of \mathcal{L} in H_δ^s . Now this index could be rather hard to compute, but because \mathcal{L} is self-adjoint on $L^2 = H_0^0$, we can identify the kernel of \mathcal{L} on H_δ^s with the cokernel of \mathcal{L} on $H_{-\delta}^s$ and the cokernel on H_δ^s with the kernel on $H_{-\delta}^s$. Hence the index of \mathcal{L} on H_δ^s agrees with one half of the ‘relative index’, i.e. the difference between the index on H_δ^s and the index on $H_{-\delta}^s$. This is fortunate because relative indices are much easier to compute since they only depend on the asymptotic geometry. In particular, it is proved in [MPU1] that

$$\text{rel-ind}(\mathcal{L})(\delta, -\delta) = \dim W \quad \text{and therefore} \quad \dim \mathcal{B} = \frac{1}{2} \dim W.$$

In case **I**, the bounded nullspace \mathcal{B} is of dimension $(1/2)(2k) = k$. In case **II** it gives $(1/2)(6k) = 3k$, and again, 6 dimensions are accounted for by rigid motions (since these surfaces are not required to have horizontal ends as in case **I**), so we get a dimension count of $3k - 6$ for the moduli space. Finally, in case **III** it gives the dimension $(1/2)(2k) = k$.

The proof that in a neighborhood of a degenerate point the moduli space is still real analytic uses an old method due to Ljapunov and Schmidt, and Kuranishi. The point is that (3.1) is not surjective, so we may not apply the implicit function theorem directly. We may modify (3.1) to get a surjective linearization as follows. Let \mathcal{D} denote the finite dimensional ‘decaying’ nullspace of \mathcal{L} , i.e. the nullspace in $H_{-\delta}^s$. Then

$$\begin{aligned} \tilde{\mathcal{L}} : H_{-\delta}^{s+2} \oplus W \oplus \mathcal{D} &\longrightarrow H_{-\delta}^s, \\ \tilde{\mathcal{L}}(v, w, \phi) &= \mathcal{L}(v + w) + \phi \end{aligned}$$

is surjective. Using this one may produce the real analytic map F between finite dimensional subspaces. This is discussed in detail in [KMP], and the version there is easily adaptable to cases **I** and **III**. The earlier proof of this in [MPU1] for case **III** uses a different sort of argument which doesn’t generalize to the other cases. Pérez and Ros do not prove this real analyticity around degenerate points in $\mathcal{M}_{\text{Min},k}$ explicitly, but note that it follows from the Weierstrass representation.

One crucial point which this general theory does not help address is whether there are *any* nondegenerate points in the moduli space. Indeed, it is conceivable that \mathcal{M}_k could consist of a single degenerate point. It is also not clear whether there are any degenerate points. It seems very difficult to give geometric criteria which ensure the nondegeneracy or degeneracy of a given solution, and it would be very interesting to make progress on this, or give examples of degenerate or nondegenerate solutions. In case **I**, Pérez and Ros note several instances where it is known that a complete embedded minimal surface is nondegenerate. This is because for certain minimal surfaces it is possible to count the dimension of all Jacobi fields; if this count agrees with the formal

dimension of the moduli space, then there can be no decaying Jacobi fields, and hence the surface is nondegenerate. In the other two cases it is plausible that any solution that is constructed explicitly, e.g. by gluing, may be checked explicitly to see if it is nondegenerate. Unfortunately, it is not at all clear whether the solutions for case **II** produced by Kapouleas [K1] or for case **III** by Schoen [S1] are nondegenerate. It is hoped that one might show that solutions for which all necksizes are sufficiently small (as is true for both Kapouleas' and Schoen's solutions) are nondegenerate. One scheme for verifying this in case **III** is presented at the end of [MPU1], but whether this scheme works is also unclear. This was one motivation for the construction in [MPa2]. The solutions with isolated singularities constructed here are nondegenerate, and this shows that the component of $\mathcal{M}_{\text{CPSC},k}$ containing these solutions is of the predicted dimension. We call these components nondegenerate. In fact, because this moduli space is real analytic, it admits a stratification into smooth real analytic manifolds. The existence of a nondegenerate solution in a given component means that this solution lies in the top dimensional 'large' stratum of that component. Unfortunately, it is not necessarily true in the real analytic category that this component is dense, but at least in the sense of measure we can then assert that generic elements in this component are nondegenerate.

The solutions in [MPU2] obtained by gluing together Delaunay metrics are also nondegenerate. Since these are conformally flat metrics, they may be uniformized and written as a singular Yamabe metric on some set $\mathbb{S}^n \setminus \Lambda$. Not every configuration Λ can be obtained this way. Of course, any Λ here must contain an even number of elements. More significantly, these configurations depend strongly on the sizes of the necks in the connected sum. In fact, as these bridging necksizes shrink, these configurations may be written as a union of pairs of points. Each pair is widely separated from any other pair, and the two points within each pair tend toward one another as the necksizes shrink. For this reason we call these solutions obtained by gluing together Delaunay metrics 'dipole metrics', or dipole solutions of case **III**. We refer to the singular sets Λ which can be obtained in this way as 'dipole configurations'. Thus, at least one component of the moduli space \mathcal{M}_Λ for any dipole configuration is nondegenerate.

To use these dipole metrics more effectively, we may introduce the unmarked moduli space for case **III**. This space is actually a closer analogue to the moduli space for case **II** treated in [KMP] than the space \mathcal{M}_Λ , which would correspond to the submoduli space of $\mathcal{M}_{\text{CMC},k}$ obtained by fixing the asymptotic axes directions. The unmarked moduli space \mathbb{M}_k is defined as the set of all complete conformally flat CPSC metrics on $\mathbb{S}^n \setminus \Lambda$ where Λ is *any* configuration of k points on \mathbb{S}^n . Using modifications of the above arguments, we show in [MPU2], that \mathbb{M}_k is a locally real analytic set of dimension $k(n+1)$, and there is a natural real analytic map

$$(3.4) \quad \pi : \mathbb{M}_k \rightarrow \mathcal{C}_k$$

onto the kn -dimensional configuration space of k distinct points on \mathbb{S}^n ; \mathcal{C}_k is naturally identified with an open subset of $(\mathbb{S}^n)^k$, and hence is a real analytic manifold. We may once again talk about nondegenerate components of \mathbb{M}_k , and we see from the

discussion above that any component of this unmarked moduli space which contains a dipole metric is nondegenerate. In particular, we obtain nondegeneracy of metrics for configurations Λ which are not necessarily dipole configurations.

The space \mathbb{M}_k is a natural setting for a more global investigation of the geometry of these moduli spaces. This is the subject of recent work of the second author [P3] which we describe below. We first recall the natural invariants associated with the elements of \mathcal{M}_Λ and the infinitesimal Lagrangian structure which these moduli spaces carry as a direct consequence of the linear analysis.

POHOŽAEV INVARIANTS

There are some interesting invariants associated to a metric $g \in \mathcal{M}_\Lambda$ at each of the singular points $p_i \in \Lambda$. These are derived from the generalized Pohožaev identity derived by Schoen [S1] which takes the form

$$\int_{\Omega} L_X R(g) dv_g = \frac{2n}{n-2} \int_{\partial\Omega} (Ric_g - n^{-1}R(g)g)(X, \nu) d\sigma_g,$$

where Ω is a domain with smooth boundary $\partial\Omega$, X is a conformal Killing vector field on Ω , L_X denotes the Lie derivative, and ν is the outward unit normal vector to $\partial\Omega$. When X is not conformal Killing one has an additional interior term. This identity is a Riemannian version of Pohožaev's original identity [Po], which in turn is a nonlinear version of the classic Rellich identity [Rel]. These are all derived from the divergence theorem.

Since any $g \in \mathcal{M}_\Lambda$ has constant scalar curvature $R(g) = n(n-1)$, this identity implies that the expression

$$\frac{2n}{n-2} \int_{\Sigma} (Ric_g - n^{-1}R(g)g)(X, \nu) d\sigma_g$$

only depends on the homology class of Σ in $H_{n-1}(\mathbb{S}^n \setminus \Lambda; \mathbb{R})$. Fix submanifolds Σ_i which are the boundaries of small balls around each singular point $p_i \in \Lambda$. Their homology classes $[\Sigma_i]$ constitute a basis for this space. Representing the space of conformal Killing fields as the Lie algebra of the conformal group $SO(n+1, 1)$, we obtain elements of the dual Lie algebra defined by

$$\mathcal{P}_i(g) = \int_{\Sigma_i} (Ric_g - (n-1)g)(\cdot, \nu) d\sigma_g \in \mathfrak{so}^*(n+1, 1).$$

These invariants play a central role in the natural compactification of the moduli space described below. It is sometimes sufficient to work with the simpler *dilatational Pohožaev invariants* defined by

$$\mathcal{P}_i^0(g) = \mathcal{P}_i(g)(X_{p_i}) \in \mathbb{R},$$

where $X_{p_i} \in \mathfrak{so}(n+1, 1)$ is the generator for the one parameter family of centered dilations fixing p_i and its antipodal point. This component of the Pohožaev invariant

may be explicitly computed from the asymptotics. This is done in [P2] and one finds that

$$\mathcal{P}_i^0(g) = c(n)H(g_{\epsilon_i})$$

where $c(n)$ is an explicit positive constant and $H(g_{\epsilon_i}) < 0$ is the Hamiltonian energy of the asymptotic Delaunay metric, as in (2.7).

The advantage of working in \mathbb{M}_k is that there is a natural action of the conformal group on this space, $SO(n+1, 1) : \mathbb{M}_k \rightarrow \mathbb{M}_k$, via

$$(g, \Lambda) \mapsto (G^*(g), G^{-1}(\Lambda)),$$

for $G \in SO(n+1, 1)$. Under this action the Pohožaev invariants transform under the co-adjoint representation of $SO(n+1, 1)$.

There are analogous invariants in case II. Suppose $\Sigma \in \mathcal{M}_{\text{CMC}, k}$ and let $\Gamma \subset \Sigma$ be a smooth 1-cycle, and $K \subset \mathbb{R}^2$ a smooth 2-chain such that $\partial K = \Gamma$. If ν is a unit normal to K and η is a unit conormal to Γ on Σ so that all corresponding orientations are coherent with that of \mathbb{R}^3 , then for any Killing vector field X in \mathbb{R}^3 , the expression

$$\mu([\Gamma])(X) = \int_{\Gamma} \eta \cdot X \, ds + \int_K \nu \cdot X \, d\sigma$$

depends only on the the homology class of Γ in Σ . This is again a clever application of the divergence theorem coupled with the first variation formula for Σ , c.f. [KKS]. Again, by choosing a basis for the first homology of Σ one has an invariant $\mu_0 \in \mathfrak{so}^*(3)$. It is easy to see that this invariant also transforms under the action of $SO(3)$ by the co-adjoint representation. This invariant is exploited in [KK] to obtain a priori area and curvature bounds for $\Sigma \in \mathcal{M}_{\text{CMC}, k}$.

COORDINATES ON THE MODULI SPACE

Here we describe one way of obtaining infinitesimal coordinates on the moduli space and recall from [MPU1] and [KMP] how these give rise to an infinitesimal Lagrangian structure. For simplicity we describe the results here for \mathcal{M}_{Λ} , though there are exact analogues for $\mathcal{M}_{\text{CMC}, k}$ and \mathbb{M}_k . We also describe the Lagrangian structure of Pérez and Ros for $\mathcal{M}_{\text{Min}, k}$.

Suppose $g \in \mathcal{M}_{\Lambda}$ is nondegenerate, so that in particular (2.10) holds. The tangent space $T_g \mathcal{M}_{\Lambda}$ is identified with the kernel of this map which we have called the ‘bounded nullspace’ $\mathcal{B} \subset H_{-\delta}^{s+2} \oplus W$. Thus any $\phi \in T_g \mathcal{M}_{\Lambda}$ can be identified (up to an exponentially decaying error) with a linear combination of elements of W ; $\phi \sim \sum_{j=1}^k (a_j(\Phi_0^+)^{(j)} + b_j(\Phi_0^-)^{(j)})$. This determines a map

$$\begin{aligned} S : T_g \mathcal{M}_{\Lambda} &\longrightarrow \mathbb{R}^{2k} \\ \phi &\longmapsto (a_1, b_1, \dots, a_k, b_k). \end{aligned}$$

The assumption that g is nondegenerate implies that S is injective: for, if $S(\phi) = 0$ then $\phi \in H_{-\delta}^{s+2}$, which would imply that $\phi = 0$, since $\mathcal{L}\phi = 0$ and g satisfies (2.8). Hence

in this case the image of S is isomorphic to the nullspace of \mathcal{L} , which is identified with $T_g\mathcal{M}_\Lambda$. Using Green's theorem [MPU1] one shows that this image is a Lagrangian subspace of \mathbb{R}^{2k} endowed with the standard symplectic form $\sum_{j=1}^k da_j \wedge db_j$. Hence a neighbourhood \mathcal{U} of g in \mathcal{M}_Λ inherits these 'Lagrangian coordinates'. This formalism works nearly identically for $\mathcal{M}_{\text{CMC},k}$ and \mathbb{M}_k . However, it is not obvious how to extend this infinitesimal Lagrangian structure to a global one. This is discussed in the next subsection.

As we described at the beginning of this section the geometric parameters for elements of $\mathcal{M}_{\text{Min},k}$ are the logarithmic growth and height parameters. For $\Sigma \in \mathcal{M}_{\text{Min},k}$ we let $(\mathbf{L}(\Sigma), \mathbf{H}(\Sigma))$ denote this $2k$ -tuple. There is a natural map $f : \mathcal{M}_{\text{Min},k} \rightarrow \mathbb{R}^{2k}$ given by $\Sigma \mapsto (\mathbf{L}(\Sigma), \mathbf{H}(\Sigma))$. One may again endow \mathbb{R}^{2k} with the standard symplectic structure as above, so that the logarithmic growth and height parameters correspond to the numbers a and b , respectively. Pérez and Ros show that the restriction of this map f to the nondegenerate surfaces is an analytic Lagrangian immersion. The fact that the map is isotropic follows from Green's theorem.

GLOBAL FEATURES OF THE MODULI SPACE

As a means towards understanding the set of all solutions to these problems more thoroughly, we would like to explore which natural geometric structures, if any, these moduli spaces carry. One immediate remark is that these spaces do not support a natural L^2 -metric because the Jacobi fields are not square integrable. Thus these spaces do not have the natural Riemannian structure which is common to many other geometric moduli spaces. Another geometric structure often found on moduli spaces is a realization of the space as a Lagrangian submanifold in a natural symplectic configuration space. The existence of such a structure for cases **II** and **III** is hinted at in the infinitesimal Lagrangian structure described above. In this subsection we describe some recent work [P3] which attempts to extend this to a global picture. Pérez and Ros also exhibit such a structure on $\mathcal{M}_{\text{Min},k}$.

The relevant configuration spaces for cases **II** and **III** are simply the spaces of Delaunay surfaces or metrics. For simplicity we discuss only case **III** here. In this context, we regard a Delaunay metric as a complete, conformally flat CPSC metric on the complement of two points in \mathbb{S}^n . Any such metric is conformally equivalent to one of the ODE solutions discussed above. These ODE solutions may be parametrized by a noncompact surface Ω with coordinates corresponding to the Delaunay and translation parameters, and corresponding area form $\omega_0 = da \wedge db$ as above. To specify a Delaunay metric we need to choose two distinct points in \mathbb{S}^n and an ODE solution. Thus the space of Delaunay metrics is simply

$$\mathcal{D} = [(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta] \times \Omega,$$

where Δ denotes the diagonal. The space of pairs of distinct points in the n -sphere, $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$, has a natural symplectic structure when regarded as the space of geodesics

in hyperbolic $(n + 1)$ -space, \mathbb{H}^{n+1} . The space of k Delaunay metrics is thus the k -fold Cartesian power

$$\mathcal{D}^k = \prod_{i=1}^k [(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta]_i \times \Omega_i.$$

\mathcal{D}^k inherits a natural symplectic structure from the product structure, ω on \mathcal{D} , coming from the symplectic structure on $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$ and the area form ω_0 on Ω . We let ω^k denote this symplectic form on \mathcal{D}^k .

There are two natural group actions on \mathcal{D}^k . The first is an action of the conformal group $S0(n + 1, 1)$ acting as hyperbolic isometries on the space of geodesics and as, suitably interpreted, conformal transformations on Ω . The second is the obvious action of the symmetric group on k letters, \mathbf{S}_k , permuting the factors of \mathcal{D}^k . Each of these groups act as symplectomorphisms on \mathcal{D}^k .

The space \mathcal{D}^k is the natural configuration space for the unmarked moduli space \mathbb{M}_k . In particular

Theorem [P3]. *There is a natural $S0(n + 1, 1) \times \mathbf{S}_k$ equivariant (possibly singular) immersion*

$$\Psi : \mathbb{M}_k \longrightarrow \mathcal{D}^k.$$

Moreover, the map Ψ is well defined on all of \mathbb{M}_k and nonsingular off of the singular locus.

This map is constructed using the Pohožaev invariants discussed above. We expect that an equivalent construction can be given analytically by using a more refined asymptotic expansion for $(g, \Lambda) \in \mathbb{M}_k$.

One should note that the dimension of \mathcal{D}^k is $2k(n + 1)$, which is twice the dimension \mathbb{M}_k . The claim is that Ψ is an equivariant Lagrangian immersion. Recall that for $\Lambda \in \mathcal{C}_k$, $\mathcal{M}_\Lambda = \pi^{-1}(\Lambda)$, where $\pi : \mathbb{M}_k \rightarrow \mathcal{C}_k$ is the projection. Let $i : \mathcal{M}_\Lambda \hookrightarrow \mathbb{M}_k$ be the inclusion. The map Ψ immediately provides a generalization of the infinitesimal Lagrangian structure on $T_g\mathcal{M}_\Lambda$ described above, namely

Corollary. *The map $\psi = (\Psi \circ i) : \mathcal{M}_\Lambda \rightarrow \mathcal{D}^k$ is isotropic, i.e. $(\psi)^*(\omega^k) = 0$.*

The expectation is that the equivariant Lagrangian immersion of \mathbb{M}_k will provide additional tools to address global questions about the moduli space, in particular the structure of its singular locus. Whether this map is actually an embedding is a difficult open question. This is related to the question of the uniqueness of elements of \mathbb{M}_k with specified asymptotics. There is an analogous picture for case **II** which will be explored elsewhere. As pointed out earlier, Pérez and Ros define a Lagrangian immersion of $\mathcal{M}_{\text{Min},k}$ into \mathbb{R}^{2k} , the space of Log and Height parameters with standard symplectic structure, in [PR]. They also compute the second fundamental form of the image of this immersion.

We have already alluded on several occasions to ways in which sequences of elements in either $\mathcal{M}_{\text{CMC},k}$ or \mathcal{M}_Λ can degenerate. These examples indicate that these moduli spaces are not compact. There are also good models for degenerations of sequences of elements in $\mathcal{M}_{\text{Min},k}$. We are thus led to the issue of whether any of these spaces admit natural compactifications. We have studied this question in some detail for the spaces \mathcal{M}_Λ and we describe these results below. Similar results should hold for the moduli spaces of CMC surfaces; this has been partially explored in [KK], but we leave the development and application of these for elsewhere. We first briefly describe the results of Ros [Ros] in this direction for $\mathcal{M}_{\text{Min},k}$.

Ros examines two types of compactness. The first he calls weak compactness, and it really describes a compactification of the moduli spaces $\mathcal{M}_{\text{Min},g,k}$. He shows that if Σ_j is a sequence of elements in one of these moduli spaces, then a subsequence converges to a finite union $\Sigma^{(1)}, \dots, \Sigma^{(\ell)}$ of minimal surfaces with lower genus or fewer number of ends. He also shows that some of his moduli spaces are ‘strongly compact’, i.e. that any sequence Σ_j as above has a subsequence converging to a minimal surface $\Sigma^{(0)}$ in the same moduli space. The criterion to show that some $\mathcal{M}_{\text{Min},g,k}$ is strongly compact is simply that all moduli spaces of surfaces with lower genus or fewer number of ends are empty. There are several known ‘nonexistence’ results of this type known. In any case, this strong compactness is a simple corollary of the more general weak compactness result.

The results for cases **II** and **III** are quite similar in form, although we never obtain actual compactness of any of our moduli spaces, simply because it is known that all the moduli spaces are nonempty. Anyway, we proceed to describe our results, concentrating on case **III**.

The first basic result gives a geometric criterion for the convergence of a sequence of elements $\{g_j\} \subset \mathcal{M}_\Lambda$.

Theorem [P2]. *Suppose that $\{g_j\} \subset \mathcal{M}_\Lambda$ is a sequence of complete CPSC metrics on $\mathbb{S}^n \setminus \Lambda$. Suppose also that the dilational Pohožaev invariants at each singular point p_i , $\mathcal{P}_i^0(g_j)$ are bounded away from zero. Then a subsequence of the g_j ’s converge uniformly in C^∞ on compact subsets of $\mathbb{S}^n \setminus \Lambda$ to an element $g_\infty \in \mathcal{M}_\Lambda$.*

A corollary of this result is that the only way in which the sequence $\{g_j\}$ can degenerate to the ‘boundary’ of the moduli space is if at some nontrivial subset $\Lambda' \subset \Lambda$ the necksizes of these metrics tend to zero. Suppose then that this occurs. The results of [P2] still imply convergence of the metric as a symmetric 2-tensor away from Λ . If $\Lambda' \neq \Lambda$, then the limiting tensor g_∞ is a nondegenerate metric away from Λ . However, one can show that g_∞ is smooth and nondegenerate across the points of Λ' ; hence it is a singular Yamabe metric on $\mathbb{S}^n \setminus (\Lambda'')$, where $\Lambda'' = \Lambda \setminus \Lambda'$. Thus g_∞ is an element of $\mathcal{M}_{\Lambda''}$. When $\Lambda' = \Lambda$ then one of two possibilities can occur: either g_∞ is isometric to the standard metric on \mathbb{S}^n , or else it vanishes identically. An illustration of this last case is provided by the explicit picture available for \mathcal{M}_Λ when $\Lambda = \{p, q\}$ consists of two points. This space can be identified with the interior of the set $\{H < 0\}$, where the

Hamiltonian energy is negative. The natural compactification of this set is obtained by adding the boundary $\{H = 0\}$, which is the union of two orbits: the zero orbit and the one corresponding to the standard metric on the sphere. So we have actually added on infinitely many copies of the spherical metric, one for each conformal dilation fixing the two singular points, but only one copy of the zero solution. The compactified moduli space $\{H \leq 0\}$ is a real subanalytic set.

Since we have given a description of all possible degenerations of sequences in \mathcal{M}_Λ , we have in fact shown that there is a compactification $\overline{\mathcal{M}}_\Lambda$ obtained by adding on certain components of the moduli spaces for fewer numbers of singular points. Exactly which of these ‘submoduli spaces’ occur in the compactification seems to be a very subtle question. In particular, it is not clear what the Hausdorff codimension of the full boundary $\overline{\mathcal{M}}_\Lambda \setminus \mathcal{M}_\Lambda$ is; when $k = 2$ this codimension is only one. It is also an interesting question whether these compactified moduli spaces are real subanalytic sets in general. It seems likely that this might be resolved via the Lagrangian embedding picture described above.

We may also consider the compactifications of the unmarked moduli spaces \mathbb{M}_k . As before, these compactifications may be analyzed by studying sequences of metrics in \mathbb{M}_k . The first result, also proved in [P2], is

Proposition 3.5. *Let $\{g_j\}$ be a sequence of elements in \mathbb{M}_k such that all dilational Pohožaev invariants at the k varying singular points $p_i^j = p_i(g_j)$ are bounded away from zero, and the distances, $d_j = \min_{i \neq k}(\text{dist}(p_i^j, p_k^j))$, between these singular points are uniformly bounded away from zero. Then a subsequence of the g_j ’s converges to an element in \mathbb{M}_k .*

There are now more possibilities to consider when analyzing the possible modes of degeneration for sequences $\{g_j\} \subset \mathbb{M}_k$. The first is that some of the dilational Pohožaev invariants tend to zero, but the singular points stay uniformly bounded away from each other. The analysis of this case is exactly the same as when the singular points are fixed. The new possibility is when some of the singular points tend to one another, and this divides into subcases, according to whether some of the dilational Pohožaev invariants tend to zero or not, and whether the singular points where these necksizes are shrinking to zero are amongst the ones colliding or not.

Suppose first that some collection of the points are coming together, but all dilational Pohožaev invariants are bounded away from zero. To be specific, suppose that the points p_1, \dots, p_ℓ stay uniformly bounded away from each other, as well as from the remaining points $p_{\ell+1}, \dots, p_k$, and suppose that these remaining points converge to some point q .

We first note that we may always assume that $\ell > 1$. To arrange this, pull back the configuration by a conformal transformation so that p_1 and p_2 are antipodal. If both of these points remain a certain distance from the others, then $\ell \geq 2$. If not, then there is a cluster converging to p_1 say. We can pull this cluster apart by a further conformal transformation, positioning p_1 and some other p_i antipodally. This process must terminate after finitely many steps, and by a relabeling of the points we are in

the situation above.

Since the dilational Pohožaev invariants at all the points remain bounded away from zero, we may extract a convergent subsequence of the metrics which tend to a limiting smooth metric g away from $\{p_1, \dots, p_\ell, q\}$. Because g has isolated singular points, at each point it is either asymptotically Delaunay or the singularity is removable, i.e. the metric extends smoothly across the point. The former occurs at the p_i because the dilational Pohožaev invariants did not tend to zero, while at q either case may occur. If g extends smoothly across q , we may picture the degeneration of the whole sequence as a ‘bubbling off’ of a singular Yamabe metric with ℓ singularities from a possibly degenerating sequence of singular Yamabe metrics with $k - \ell$ singularities. If this sequence with $k - \ell$ singularities does not itself degenerate, then the whole degeneration corresponds to the shrinking of a neck in a connected sum between two singular Yamabe metrics, one with ℓ singularities and one with $k - \ell$ singularities. However, even if the solution with $k - \ell$ singularities degenerates, because all dilational Pohožaev invariants are bounded away from zero, we can repeat this analysis. The other case, when g is singular at q , may be pictured as the degeneration of a sequence of connected sums between two metrics g_α and g_β , where g_α converges to an element of $\mathbb{M}_{\ell+1}$, but g_β may possibly degenerate further, and where these metrics are connected at points x_α, x_β , where x_α converges to one of the singularities of the limit of g_α . In any event, we may still isolate the $k - \ell$ singular points by pulling back with a conformal transformation and continue the analysis. In the end, we obtain the

Proposition 3.6. *Suppose that $\{g_j\}$ is any sequence of elements in \mathbb{M}_k , such that the dilational Pohožaev invariants at the singular points p_i^j are all bounded away from zero. Then there is a partition of $\{1, \dots, k\}$ into subsets I_1, \dots, I_s , with each $|I_\ell| \geq 2$, such that for each I_ℓ there is a sequence of conformal transformations f_j such that a subsequence of $f_j^*(g_j)$ converges to an element of $\mathbb{M}_{|I_\ell|}$.*

Thus the general picture in this case is of s different singular Yamabe metrics bridged together by connected sums, with all connecting necks shrinking to zero in the limit. The construction in [MPU2] can be used to give examples of this multi-connected sum degeneration.

When some of the dilational Pohožaev invariants tends to zero, it may seem that there are further cases to consider. However, a moment’s thought shows that, using the results of [P2], the only difference in this case is that in the preceding proposition, having chosen I_ℓ and the conformal transformations f_j , the limits of the metrics $f_j^*g_j$ converge to elements in some unmarked moduli space \mathbb{M}_m where $m \leq |I_\ell|$. Thus the general statement of this proposition may be altered to cover all cases by simply requiring $\{I_1, \dots, I_s\}$ to be a partition of the subset $\{p_{i_1}, \dots, p_{i_r}\}$ of points where the dilational Pohožaev invariants do not tend to zero.

This completes the analysis of the possible degenerations of sequences in \mathbb{M}_k . The conclusion is that a compactification $\bar{\mathbb{M}}_k$ may be obtained by adding on a union of certain subsets of unmarked moduli spaces of metrics singular at fewer than k points.

REFERENCES

- [Ab1] U. Abresch, *Constant mean curvature tori in terms of elliptic functions*, J. Reine Angew. Math. **374** (1987), 169-192.
- [Ab2] U. Abresch, *Old and new periodic solutions of the sinh-Gordon equation*, Seminar on new results in non-linear partial differential equations (1987), Vieweg.
- [Al] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large*, V. Vestnik Leningrad Univ. Mat. Mekh. Astronom, **13** (1958), 5-8; Amer. Math. Soc. Transl. Ser.2 **21**, 412-416.
- [AKS] P. Aviles, N. Korevaar & R. Schoen, *The symmetry of constant scalar curvature metrics near point singularities*, unpublished.
- [B] A.I. Bobenko, *All constant mean curvature tori in $\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3$ in terms of theta-functions*, Math. Ann. **290** (1991), 209-245.
- [CGS] L. Caffarelli, B. Gidas & J. Spruck, *Asymptotic Symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), 271-297.
- [CL1] C.C. Chen & C.S. Lin, *Local behavior of singular positive solutions of semilinear elliptic equations with Sobolev exponent*, Duke Math. J. **78** (1995), 315-334.
- [CL2] C.C. Chen & C.S. Lin, *A simple proof of a Theorem of Caffarelli-Gidas-Spruck*, Preprint.
- [Co] C. Costa, *Example of a complete minimal immersion in \mathbb{R}^3 of genus one and three embedded ends*, Bull. Soc. Bras. Mat. **15** (1984), 47-54.
- [D] C. Delaunay, *Sur la surface de revolution dont la courbure moyenne est constant*, J. Math. Pure Appl. **6** (1841), 309-320.
- [E] J. Eells, *The surfaces of Delaunay*, Math. Intelligencer **9** (1987), 53-57.
- [F] D.L. Finn, *Positive solutions to nonlinear elliptic equations with prescribed singularities*, Ph.D. Dissertation, Northeastern University (1995).
- [FMc] D.L. Finn & R.C. McOwen, *Singularities and asymptotics for the equation $\Delta_g u - u^q = Su$* , Indiana Univ. Math. J. **42** (1993), 1487-1523.
- [Fo1] R. H. Fowler, *The form near infinity of real continuous solutions of a certain differential equation of the second order*, Quarterly J. of Pure and Appl. Math. **45** (1914), 289-349.
- [Fo2] R. H. Fowler, *Further studies of Emden's and similar differential equations*, Quarterly J. of Math., Oxford series **2** (1931), 259-287.
- [G] K. Grosse-Brauckmann, *New surfaces of constant mean curvature*, Math. Z. **214** (1993), 527-565.
- [GK] K. Grosse-Brauckmann & R. Kusner, *On the moduli spaces of embedded constant mean curvature surfaces with three or four ends*, In preparation.
- [HKa] D. Hoffman & H. Karcher, *Complete embedded minimal surfaces of finite total curvature*, MSRI Preprint No. 064-95.
- [HM] D. Hoffman & W. Meeks III, *A complete embedded minimal surface in \mathbb{R}^3 with genus 1 and three ends*, J. Differential Geometry **21** (1985), 109-127.
- [Ka1] N. Kapouleas, *Complete constant mean curvature surfaces in Euclidean three-space*, Ann. of Math. **131** (1990), 239-330.
- [Ka2] N. Kapouleas, *Compact constant mean curvature surfaces in Euclidean three-space*, J. Differential Geometry **33** (1990), 683-715.
- [Ka3] N. Kapouleas, *Constant mean curvature surfaces constructed by fusing Wente tori*, Invent. Math. **119** (1995), 443-518.
- [KaMP] N. Kapouleas, R. Mazzeo & D. Pollack, In preparation.
- [KK] N. Korevaar & R. Kusner, *The global structure of constant mean curvature surfaces*, Invent. Math. **114** (1993), 311-332.
- [KKS] N. Korevaar, R. Kusner & B. Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geometry **30** (1989), 465-503.
- [KMP] R. Kusner, R. Mazzeo & D. Pollack, *The moduli space of complete embedded constant mean curvature surfaces*, To appear Geom. and Funct. Analysis.

- [LN] C. Loewner & L. Nirenberg, *Partial differential equations invariant under conformal or projective transformations*, Contributions to Analysis, Academic Press, New York, 1974, pp. 245-272.
- [M1] R. Mazzeo, *Regularity for the singular Yamabe equation*, Indiana Univ. Math. J. **40** (1991), 1277-1299.
- [M2] R. Mazzeo, *Elliptic theory of differential edge operators I*, Comm. in P.D.E. **16** (1991), 1615-1664.
- [MPa1] R. Mazzeo & F. Pacard, *A new construction of singular solutions for a semilinear elliptic equation*, To appear, J. Differential Geometry.
- [MPa2] R. Mazzeo & F. Pacard, *Singular Yamabe metrics with isolated singularities*, In preparation.
- [MPU1] R. Mazzeo, D. Pollack & K. Uhlenbeck, *Moduli spaces of singular Yamabe metrics*, To appear, J. Amer. Math. Soc.
- [MPU2] R. Mazzeo, D. Pollack & K. Uhlenbeck, *Connected sum constructions for constant scalar curvature metrics*, To appear, Top. Methods in Nonlinear Anal.
- [MS] R. Mazzeo & N. Smale, *Conformally flat metrics of constant positive scalar curvature on subdomains of the sphere*, J. Differential Geometry **34** (1991), 581-621.
- [Mc] R. McOwen, *Singularities and the conformal scalar curvature equation*, Geometric Analysis and Nonlinear PDE (I. Bakelman, eds.), Marcel Dekker, 1992.
- [Mee] W. Meeks III, *The topology and geometry of embedded surfaces of constant mean curvature*, J. Differential Geometry **27** (1988), 539-552.
- [Mel] R. Melrose, *The Atiyah-Patodi-Singer index theorem*, AK Peters Ltd., Wellesley, MA, 1993.
- [O] R. Osserman, *A survey of minimal surfaces*, Dover Publ. Inc., New York, 1989.
- [PR] J. Pérez & A. Ros, *The space of properly embedded minimal surfaces with finite total curvature*, preprint.
- [Pa] F. Pacard, *The Yamabe problem on subdomains of even dimensional spheres*, To appear, Top. Methods in Nonlinear Anal.
- [PS] U. Pinkall & I. Sterling, *On the classification of constant mean curvature tori*, Ann. of Math. **130** (1989), 407-451.
- [Po] S. Pohožaev, *Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$* , Soviet Math. Doklady **6** (1965), 1408-1411.
- [P1] D. Pollack, *Nonuniqueness and high energy solutions for a conformally invariant scalar equation*, Comm. Anal. and Geom. **1** (1993), 347-414.
- [P2] D. Pollack, *Compactness results for complete metrics of constant positive scalar curvature on subdomains of \mathbb{S}^n* , Indiana Univ. Math. J. **42** (1993), 1441-1456.
- [P3] D. Pollack, *Geometric moduli spaces as Lagrangian submanifolds*, In preparation.
- [Reb] Rebaï, *Weak solutions of nonlinear elliptic equations with prescribed singular set*, To appear, J. Differential Equations.
- [Rel] F. Rellich, *Darstellung der Eigenwerte von $\Delta u + \lambda u$ durch ein Randintegral*, Math. Z. **46** (1940), 635-646.
- [Ros] A. Ros, *Compactness of the space of properly embedded minimal surfaces with finite total curvature*, Indiana Univ. Math. J. **44** (1995), 139-152.
- [S1] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geometry **20** (1984), 479-495.
- [S2] R. Schoen, *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation*, Comm. Pure and Appl. Math. **XLI** (1988), 317-392.
- [S3] R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, Topics in Calculus of Variations; Lecture Notes in Mathematics # 1365 (M. Giaquinta, eds.), Springer-Verlag, 1987, pp. 120-154.
- [S4] R. Schoen, *A report on some recent progress on nonlinear problems in geometry*, Surveys in Differential Geometry; supplement to J. Differential Geometry **1** (1991), 201-242.
- [SY] R. Schoen & S.T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988), 47-71.

- [T] C. Taubes, *Gauge theory on asymptotically periodic 4-manifolds*, J. Differential Geometry **25** (1987), 363-430.
- [Tr] M. Traizet, *Construction de surfaces minimales en recollant des surfaces de Scherk*, Ph.D. Dissertation, Paris VI, (1995).
- [U] K. Uhlenbeck, In preparation.
- [W1] H. Wente, *A counterexample to a conjecture of H. Hopf*, Pacific J. Math. **121** (1984), 193-243.
- [W2] H. Wente, *Twisted Tori of constant mean curvature in \mathbb{R}^3* , Seminar on new results in non-linear partial differential equations (1987), Vieweg.