Recall the definition of the Schwartz space on $\mathbb{R}^n$:

$$S = \{ f : \partial_\alpha^\alpha x^\beta f \in L^1 \forall \alpha, \beta \}.$$ 

Here $\alpha$ and $\beta$ are multiindices of nonnegative integers. This definition is slightly different than the one I gave in class since I am using the monomials $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ rather than $(1+|x|)^k$, but this is easily seen to be equivalent.

1) Let $f \in S$. Provide the details of the assertion that its Fourier transform $\hat{f}(\xi) \in S$ as well.

2) Again let $f \in S$ and consider for $t > 0$ the function

$$(H_t * f)(x) = (2\pi)^{-n} \int \int e^{i(x-y) \cdot \xi - t|\xi|^2} f(y) dyd\xi.$$ 

In class we considered the same operator (with $t$ denoted by $\epsilon$ then), and showed how to evaluate the kernel

$$H_t(z) = (2\pi)^{-n} \int e^{iz \cdot \xi - t|\xi|^2} d\xi.$$ 

Prove that $H_t * f \to f$ in the topology of $S$. In addition, show that $u(t, x) := (H_t * f)(x)$ is a smooth function on $t \geq 0$ which satisfies the heat equation $(\partial_t - \Delta)u = 0$.

3) Suppose that $f \in L^1(\mathbb{R})$ has support in the finite interval $[-A, A]$. Prove that its Fourier transform $\hat{f}(\xi)$ extends to a holomorphic function in $\zeta = \xi + i\eta$ in some strip around the $\xi$ axis. Conclude that if $f \in C_0^\infty(\mathbb{R})$, then $\hat{f}(\xi)$ cannot have compact support.