Solutions to Homework Assignment 6

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Section 4.3, Problem 5.

Let the centre $Z(G)$ of G have index n. Consider $g \in G$. Then the number of conjugates of g in G is equal to the index of $C_G(g)$ (i.e. the centraliser of g in G) in G. But by definition any centraliser contains the centre, $Z(G) \leq C_G(g)$. As $Z(G)$ has index n in G, $C_G(g)$ must have no greater index. so $C_G(g)$ has index at most n. Hence g has at most n elements. As this is true for any $g \in G$, every conjugacy class has at most n elements.

Section 4.3, Problem 19.

We have: $H \trianglelefteq G$, K is a conjugacy class of G contained in H, and $x \in K$. Note that if x, y are conjugate in H, then they are clearly conjugate in G; but the converse is not necessarily true. Thus K splits into a number of distinct conjugacy classes in H. The size of the conjugacy class of x in G is $|K| = |G : C_G(x)|$. Let K' denote the conjugacy class of x in H, so $|K'| = |H : C_H(x)|$. Now we see

$$\frac{|K|}{|K'|} = \frac{|G|}{|H|} \frac{|C_H(x)|}{|C_G(x)|}.$$

By the second isomorphism theorem we also have

$$\frac{HC_G(x)}{H} \cong \frac{C_G(x)}{C_G(x) \cap H} \cong \frac{C_G(x)}{C_H(x)} \Rightarrow \frac{|C_H(x)|}{|C_G(x)|} = \frac{|HC_G(x)|}{|H|}.$$

The last isomorphism is true since $C_G(x) \cap H$ is the set of elements in G which commute with x, which also lie in H; that is, $C_G(x) \cap H = C_H(x)$. Putting these together gives

$$\frac{|K|}{|K'|} = \frac{|G|}{|HC_G(x)|}.$$

But $H \trianglelefteq G$ and $C_G(x) \leq G$, so $HC_G(x) \leq G$ and we have $|K|/|K'| = |G : HC_G(x)|$. We only now need check that all the conjugacy classes into which K splits have the same size; a priori $|K'|$ could vary as we consider different x. So take $x, y \in K$ and we will prove that $|HC_G(x)| = |HC_G(y)|$. But by the equation above it is sufficient to show $|C_G(x)| = |C_G(y)|$ and $|C_H(x)| = |C_H(y)|$. Since $x, y \in K$ they are conjugate in G, so $y = gxg^{-1}$ for some $g \in G$. Now we see that $C_G(y) = C_G(gxg^{-1}) = gC_G(x)g^{-1}$: for $a \in C_G(x)$ means that $axa^{-1} = x$, which is equivalent to $(gag^{-1})(gxg^{-1})(ga^{-1}g^{-1}) = gxg^{-1}$, which is equivalent to $(gag^{-1})y(gag^{-1}) = y$, i.e. $gag^{-1} \in C_G(y)$. As $C_G(y)$ is conjugate to $C_G(x)$, they have the same size. Similarly we have $C_H(y)$ is conjugate to $C_H(x)$ in H, and has the same size as well. So K splits into a union of conjugacy classes in H of the same size; hence the number of conjugacy classes is $|K|/|K'| = |G : HC_G(x)|$.

Section 4.4, Problem 7.

Suppose H is the unique subgroup of a given order in G. Consider an automorphism ϕ of G. Then ϕ sends H to some subgroup of G of the same size; but as H is the only subgroup of that size, $\phi(H) = H$. Thus H is characteristic in G.

Section 4.4, Problem 8.

We have $H \leq K \leq G$.

(i) Suppose H is characteristic in K and $K \trianglelefteq G$. Consider $g \in G$; we will show $gHg^{-1} = H$, i.e. $H \trianglelefteq G$. Since $K \trianglelefteq G$ we certainly have $gKg^{-1} = K$. But now the map ϕ defined by $x \mapsto gxg^{-1}$ is an inner automorphism of G which preserves K; hence the restriction $\phi|_K$ is an automorphism of K. As H is characteristic in K, $\phi|_K(H) = H$. Thus $gHg^{-1} = H$, as desired.

(ii) Now suppose H is characteristic in K and K is characteristic in G. Consider an automorphism ϕ of G. As K is characteristic $\phi(K) = K$, so $\phi|_K$ is an automorphism of K. As H is characteristic in K, $\phi|_K(H) = H$. Hence $\phi(H) = H$. So H is characteristic in G.

We use this to prove the Klein 4-group V_4 is characteristic in S_4. For V_4 consists precisely of the identity and the elements of order 2 in A_4. Hence any automorphism of A_4 preserves V_4, so V_4 is characteristic in A_4. And A_4 is characteristic in S_4. One way to see this is to use the theorem that all automorphisms of S_4 are inner automorphisms, hence preserve the normal subgroup A_4. Another way to see this is to note that A_4 is the unique subgroup of S_4 of order 12: any such subgroup has index 2, hence is normal, hence is a union of conjugacy classes. The conjugacy classes in S_4 correspond precisely to different cycle structures: so there is one element conjugate to (1); 6 elements conjugate to (12); 3 elements conjugate to (12)(34); 8 elements conjugate to (123); and 6 elements conjugate to (1234). The only way to make these numbers sum to 12 is to take $1 + 3 + 8$, which gives the subgroup A_4.

(iii) We provide an example to show that $H \trianglelefteq K$, K characteristic in G does not imply $H \trianglelefteq G$. Let

$$H = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z} \right\}, \quad K = \left\{ \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \mid s \in \mathbb{R} \right\}, \quad G = \left\{ \begin{bmatrix} r & s \\ 0 & 1 \end{bmatrix} \mid r, s \in \mathbb{R} \right\}.$$

Checking that rules for matrix multiplication and inversion, it is easily verified that these are all groups. To see that $H \trianglelefteq K$, take an element of H and K and check

$$\begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in H.$$

To see that K is characteristic in G, consider an automorphism ϕ of G. Then ϕ preserves the elements of order 2 in G. But these are precisely the non-identity solutions of

$$\begin{bmatrix} r & s \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} r^2 & s(r+1) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus we see that the elements of order 2 are precisely those of the form

$$\begin{bmatrix} -1 & s \\ 0 & 1 \end{bmatrix}, \quad s \in \mathbb{R},$$

and ϕ takes any matrix of this form to another of this form. But now taking an element of K, we see that

$$\phi\left(\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} -1 & s+1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & u \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t-u \\ 0 & 1 \end{bmatrix} \in H$$

for some $t, u \in \mathbb{R}$, so K is characteristic in G. Finally we see that H is not normal in G; for instance

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \notin H.$$

Section 4.4, Problem 9.

Let r, s are the usual generators for D_{2n}, let $K =< r >$ be a subgroup of order n and consider a subgroup H of K. Now H is characteristic in K: K is cyclic, hence isomorphic to the additive group $\mathbb{Z}/n/\mathbb{Z}$, and H then corresponds to some subgroup $m(\mathbb{Z}/n\mathbb{Z})$ where $m|n$. An automorphism ϕ of K is multiplication by some number, say l, relatively prime to n; and $l(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. Then $\phi(H) = lm(\mathbb{Z}/n\mathbb{Z}) = m(\mathbb{Z}/n\mathbb{Z}) = H$, so H is characteristic in K. Also K is characteristic in D_{2n}: note D_{2n} contains elements of order n, such as r, all of which lie in K; in fact all elements of D_{2n} outside K have order 2 (they are reflections of the n-gon). Letting $\phi \in \mathrm{Aut}\,(D_{2n})$, we see that $\phi(r)$ lies in K, and $\phi(K) \subseteq K$; but $|\phi(K)| = |K|$, hence $\phi(K) = K$. Now using the result of part (b) above, H is normal in D_{2n}.

Section 4.5, Problem 1.

G is a finite group, P is a p-Sylow subgroup, and $P \leq H \leq G$. Let $|G| = p^a n$ where p does not divide n. Then $|P| = p^a$, so $|H|$ is a multiple of p^a and a factor of $p^a n$. Hence $|H| = p^a m$ where m is a positive integer dividing n. Thus p^a is the highest power of p dividing $|H|$ and P is a p-Sylow subgroup of H.

It is not true in general that if $P \leq H \leq G$ and P is a p-Sylow subgroup of H, then P is a p-Sylow subgroup of H. For instance suppose G has order 12, H has order 6 and P has order 2. Then P is a 2-Sylow subgroup of H but not of G. For a specific example we may take $G = \mathbb{Z}/12\mathbb{Z}$, $H = 2(\mathbb{Z}/12\mathbb{Z})$ and $P = 6(\mathbb{Z}/12\mathbb{Z})$ (all taken as additive groups).

Section 4.5, Problem 35.

P is a p-Sylow subgroup of G and $H \leq G$. Let $|H| = p^a m$ and $|G| = p^b mn$, so a, b, m, n are positive integers and $a \leq b$, and $|P| = p^b$. Let Q be a p-Sylow subgroup of H, so $|Q| = p^a$. Then Q is also a p-subgroup of G, hence by the Sylow theorem lies in some conjugate of P, say $Q \leq gPg^{-1}$. Then $gPg^{-1} \cap H$ is a subgroup of H containing Q, so its order is a multiple of $|Q| = p^a$, and a factor of both $|gPg^{-1}| = |P| = p^b$ and $|H| = p^a m$. Thus $|gPg^{-1} \cap H| = p^a$, and $gPg^{-1} \cap H = Q$, and $gPg^{-1} \cap H$ is a p-Sylow subgroup of H.

We give an explicit example where P is a p-Sylow subgroup of G, $H \leq G$, and for every $h \in H$, $hPh^{-1} \cap H$ is not a p-Sylow subgroup of H. Take $G = S_4$, $p = 3$, $P = \{(1), (123), (132)\}$ and $H = \{(1), (134), (143)\}$. Then there are three possibilities for hPh^{-1}, namely $\{(1), (123), (143)\}$, $\{(1), (324), (342)\}$ and $\{(1), (421), (412)\}$. In each case $hPh^{-1} \cap H = \{(1)\}$, which is not a 3-Sylow subgroup of H.