

# Convexity in Contact Topology

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## Contents

|           |   |           |
|-----------|---|-----------|
| <b>I</b>  | <b>The notion of convexity</b>  | <b>4</b>  |
| 1         | Preliminary definitions . . . . .                                     | 4         |
|           | A Symplectic and contact structures . . . . .                         | 4         |
|           | B Singular dimension-1 foliations . . . . .                           | 4         |
|           | C Characteristic foliation of a hypersurface . . . . .                | 4         |
| 2         | Characteristic hypersurface of a contact vector field . . . . .       | 5         |
|           | A Contact vector field . . . . .                                      | 5         |
|           | B Characteristic hypersurface . . . . .                               | 5         |
|           | C Singularities of contact vector fields . . . . .                    | 6         |
| 3         | Convex hypersurfaces . . . . .  | 7         |
|           | A Definition, example . . . . .                                       | 7         |
|           | B Vertically invariant contact structures . . . . .                   | 8         |
| 4         | Convex contact structures . . . . .                                   | 10        |
|           | A Pseudo-gradients of a Morse function . . . . .                      | 10        |
|           | B Notion and condition of convexity for a contact structure . . . . . | 11        |
|           | C Examples of convex contact structures . . . . .                     | 11        |
| <b>II</b> | <b>On the characteristic foliation of surfaces in dimension 3</b>     | <b>12</b> |
| 1         | Properties of characteristic foliations . . . . .                     | 12        |
|           | A General form of characteristic foliations . . . . .                 | 13        |
|           | B Generic properties of characteristic foliations . . . . .           | 14        |
| 2         | Convex surfaces . . . . .   | 15        |
|           | A Dividing set of a convex surface . . . . .                          | 15        |
|           | B Examples of non-convex surfaces . . . . .                           | 16        |
|           | C Examples of convex surfaces . . . . .                               | 17        |
| 3         | Deformations of characteristic foliations . . . . .                   | 18        |
|           | A A reduced form for characteristic foliations . . . . .              | 18        |
|           | B Elimination of singularities . . . . .                              | 18        |
|           | C End of the proof of proposition 3.1 . . . . .                       | 20        |
|           | D Foliations adapted to a given dividing set . . . . .                | 20        |

|   |           |
|---|-----------|
| <b>III Construction of convex contact structures in dimension 3</b> | <b>22</b> |
| 1 Convex contact structures and essential surfaces . . . . .        | 22        |
| A Existence results . . . . .                                       | 22        |
| B Scheme of the proof of theorem 1.2 . . . . .                      | 23        |
| 2 Attachment of handles of index 0 and 3 . . . . .                  | 24        |
| A The model . . . . .   | 24        |
| B Handles of index 0 . . . . .                                      | 24        |
| C Handles of index 3 . . . . .                                      | 24        |
| 3 Attachment of handles of index 1 and 2 . . . . .                  | 25        |
| A The model . . . . .   | 25        |
| B Handles of index 2 . . . . .                                      | 26        |
| C Handles of index 1 . . . . .                                      | 28        |
| <b>IV Construction of essential surfaces</b>                        | <b>28</b> |
| 1 Some examples . . . . .   | 28        |
| A Examples of essential surfaces . . . . .                          | 28        |
| B An example of a function having no essential surface (con-        |           |
| structed by C. Lescop) . . . . .                                    | 30        |
| 2 A general method for constructing essential surfaces . . . . .    | 30        |
| A Splitting along an essential surface . . . . .                    | 30        |
| B The principal construction . . . . .                              | 32        |
| C An existence theorem . . . . .                                    | 34        |

## Translator’s note

My French is not great, but with a dictionary hopefully most of this is accurate. Any doubts are written in footnotes. All footnotes are mine. Figures and references can be found in the original.

## Introduction

This article tackles the study of convexity in contact geometry, as it has been defined in [EG]: a structure, symplectic or contact, is called convex if it is conformally invariant under the gradient flow of a proper Morse function. For symplectic manifolds, this property plays the same role that strict pseudo-convexity occupies in complex analytic manifolds. It can only, for example, be shown on open manifolds having the homotopy type of polyhedra of half dimension and, in [EG], Ya. Eliashberg and M. Gromov show how it tempers it the geometry and forbids certain exotic phenomena (see also [Gr] and [E11]). In contact geometry, the situation presents itself differently. First of all, the usual structures, on jet spaces of order 1, spheres and manifolds of contact elements, are all convex (see I.4.C). Then, the results obtained here show that in dimension 3, there exist numerous convex contact manifolds. In particular, certain exotic structures discovered by T. Erlandsson and D. Bennequin (see [Be]) are convex; in fact, we do not know any examples of non-convex structures.

The approach adopted is the following: given a proper Morse function  $f$  on a 3-dimensional manifold  $V$ , one tries to construct on  $V$  a contact structure  $\xi$  which is invariant under the flow of the gradient  $X$  of  $f$ . The study of contact fields (i.e. fields preserving the contact structure) shows that, if this structure  $\xi$  exists, the surface  $C$  of points of  $V$  where  $X$  is tangent to  $\xi$  must satisfy, vis-à-vis  $f$ , the following conditions (Proposition I.4.5):

- (i)  $f|_C$  is a proper Morse function;
- (ii) the critical points of  $f$  are all on  $C$  and are exactly the critical points of  $f|_C$ ;
- (iii)  $f$  and  $f|_C$  have the same local extrema.

A Morse function does not always admit surfaces satisfying these properties (see IV.1.B). Nevertheless, one can modify it, adding only several critical points of indices 1 or 2 in a position of elimination, so that such a surface  $C$  exists (Theorem IV.2.7). Also, being given  $C$  allows one effectively to construct the desired contact structure  $\xi$  (Theorem III.1.2). To obtain this, one puts on each handle an induced structure via immersion in a well-chosen model on  $\mathbb{R}^3$ . The difficulty is to adjust these immersions in order to be able to reglue the pieces: this problem is localised along certain faces of the handles. Yet, in the neighbourhood of a surface, a contact structure is entirely described by the (singular) dimension-1 foliation that it traces on the surface. Moreover, each surface considered here, corresponding to a regular level set of  $f$ , is found, by construction, to be transverse in  $\mathbb{R}^3$  to a vector field that preserves the model structure and holds the role of gradient of  $f$ . The crucial point then is to understand how, when one moves the surface by an isotopy always transversal to this field, the foliation is modified (Proposition II.3.6). At this point, a convex contact structure appears geometrically describable by a finite number of these foliations, carried by the different regular level sets of the function and determined only up to the previous modifications.

Among these possible modifications of the foliation is the elimination of pairs of singularities (Lemma II.3.3). One can thus extend a result of Ya. Eliashberg that permits the elimination of certain complex points on a surface contained in the pseudo-convex boundary of a holomorphic domain (see [E11], Theorem 6.1 and [E12]). For this, in place of the theory of holomorphic curves in 4-dimensional symplectic manifolds, we use the following remarkable fact (Proposition II.2.6): in a contact 3-dimensional manifold, a surface generically possesses a transverse contact vector field. Thanks to this property of invariance, the problem of elimination reduces to the symplectic geometry of surfaces.

The problems studied in this article have been exposed to me by Yasha Eliashberg during some conversations which were marvellously enriching for me; I thank him heartily. I equally thank Francois Laudembach and Jean-Claude Sikorav for their numerous remarks and pertinent suggestions regarding the text.

# I The notion of convexity

## 1 Preliminary definitions

### A Symplectic and contact structures

A *symplectic structure on a vector space*  $V$  of dimension  $2n$  is an exterior 2-form  $\omega$  with exterior  $n$ 'th power nonzero. The *orthogonal complement* of a subspace  $W$  of  $V$  is the subspace  $\{v \in V \mid \forall w \in W, \omega(v, w) = 0\}$ .

One says that  $W$  is *coisotropic* if it contains its orthogonal complement. Note that, if  $c$  is a nonzero real number,  $c\omega$  is another symplectic form and the orthogonal complement of  $W$  is the same for  $\omega$  and  $c\omega$ .

A *symplectic structure on a vector bundle* of rank two is a field of symplectic forms on its fibres.

A *symplectic structure on a manifold*  $V$  of dimension  $2n$  is a closed differential 2-form  $\omega$  which induces on each tangent space a symplectic form.

A *contact structure* on a manifold  $V$  of dimension  $2n + 1$  is a completely non-integrable hyperplane field  $\xi$ , that is to say, defined locally by a 1-form  $\alpha$  such that  $\alpha \wedge (d\alpha)^n$  is non-vanishing. In other words,  $d\alpha|_\xi$  is at each point a symplectic form. Multiplying  $\alpha$  by a nowhere vanishing function  $f$  changes  $d\alpha|_\xi$  to  $f \cdot d\alpha|_\xi$ , so that  $\xi$  is provided with a *conformal symplectic structure*. We also remark that, if  $n$  is even,  $\xi$  is naturally oriented while, if  $n$  is odd,  $V$  is naturally oriented. In either case, any orientation transverse to  $\xi$  (which exists if and only if  $\xi$  admits a global equation  $\alpha = 0$ ) orients at the same time  $\xi$  and  $V$ .

### B Singular dimension-1 foliations

In this text, a *singular (dimension 1) foliation* on a manifold  $M$  of dimension  $m$  is a foliation  $\mathcal{F}$  defined by an atlas  $\{U_i, X_i\}$  where:  $\{U_i\}$  is a cover of  $M$ ,  $X_i$  a vector field on  $U_i$  and, for every  $(i, j)$ , there exists a nonvanishing function  $f_{ij}$  on  $U_i \cap U_j$  such that  $X_i = f_{ij}X_j$ .

**Remark 1.1** *If each  $U_i$  is provided with a volume form  $\theta_i$ , then being given  $X_i$  is equivalent to being given an  $(m - 1)$ -form  $i(X_i)\theta_i$  (interior product of  $\theta$  with  $X_i$ ).*

We say that a vector field  $X$  on  $M$  *directs*  $\mathcal{F}$  if, for all  $i$ , there exists a nonvanishing function  $f_i$  on  $U_i$  such that  $X = f_iX_i$ ; we say that  $\mathcal{F}$  is *orientable* if such a field exists.

### C Characteristic foliation of a hypersurface

Let  $S$  be a hypersurface in a contact manifold  $(V, \xi)$  of dimension  $2n + 1$ . The trace on  $\xi$  of the tangent bundle of  $S$  determines a distribution (of non-constant rank) of coisotropic subspaces in  $\xi|_S$ . The orthogonal distribution, via the conformal symplectic structure on  $\xi|_S$ , is of rank 0 on the singular locus  $\Sigma$  where  $\xi$  is tangent to  $S$ , and of rank 1 otherwise. It defines a singular foliation, in the sense of B, that we call the characteristic foliation of  $S$ . Locally, if  $\theta$  is a volume

form on  $S$  and  $\beta$  the 1-form induced by an equation for  $\xi$ , the characteristic foliation is defined by the vector field  $X$  such that  $i(X)\theta = \beta \wedge (d\beta)^{n-1}$ . One easily proves that the characteristic foliation of  $S$  is orientable if and only if the normal bundle of  $S$  is isomorphic to the quotient bundle  $(TV/\xi)|_S$ .

**Remark 1.2** *Outside the singular locus  $\Sigma$ , the characteristic foliation  $\mathcal{F}$  of  $S$  has a transverse contact structure,  $(\xi \cap TS)/\mathcal{F}$ , invariant under holonomy (see [McD]). On  $\Sigma$ ,  $TS|_\Sigma = \xi|_\Sigma$  has a conformal symplectic structure, invariant under local vector fields which direct  $\mathcal{F}$  (see 2.C).*

## 2 Characteristic hypersurface of a contact vector field

### A Contact vector field

Let  $(V, \xi)$  be a contact manifold.

**Definition 2.1** *A contact vector field on  $(V, \xi)$  is a vector field whose flow preserves  $\xi$ .*

It is well known (see [A]) that:

**Proposition 2.2** *The contact vector fields on  $(V, \xi)$  are in bijective correspondence with the sections of the normal bundle to  $\xi$ ,  $TV/\xi$ . In other words, any section of this quotient lifts to a unique contact vector field.*

**Corollary 2.3** *Any contact vector field given locally can be extended globally.*

*Remark.* In the presence of an equation for  $\xi$ , i.e. a trivialisation of  $TV/\xi$ , a section of  $TV/\xi$  is nothing other than a function, called Hamiltonian, on the corresponding contact vector field.

### B Characteristic hypersurface

Let  $X$  be a contact vector field on  $(V, \xi)$ .

**Definition 2.4** *The characteristic hypersurface of  $X$  is the set  $C = C(X)$  of points where  $X$  is tangent to  $\xi$ .*

On the space of vector fields (furnished with the  $C^\infty$  topology), the property of having a reduction modulo  $\xi$  transverse to the zero section of  $TV/\xi$  is generic. In this case, by abuse of language, we will say that the vector field is *generic*. Its characteristic hypersurface is then regular.

**Proposition 2.5** *If  $X$  is generic,  $X$  is tangent to its characteristic hypersurface  $C$  and directs the characteristic foliation on it.*

**PROOF** The flow of  $X$  preserves  $X$  and  $\xi$ , therefore  $C$ , so that  $X$  is tangent to  $C$ .

Now let  $x$  be a point of  $C$  and  $\alpha$  a local equation for  $\xi$  near  $x$ . The hypersurface  $C$  is defined locally by the equation  $i(X)\alpha = 0$  (regular since  $X$  is generic). Also, as  $X$  is contact, the Lie derivative of  $\alpha$  satisfies:  $L_X\alpha = g\alpha$  for some function  $g$ . For  $v \in T_x C \cap \xi_x$ , we then have:

$$\begin{aligned} d\alpha(x) \cdot (X(x), v) &= (L_X\alpha)(x) \cdot v - (di(X)\alpha)(x) \cdot v \\ &= (g\alpha)(x) \cdot v - (di(X)\alpha)(x) \cdot v = 0 \end{aligned}$$

since the two terms are zero. Thus  $X(x)$  is orthogonal to  $T_x C \cap \xi_x$ .

Moreover, if  $X(x) = 0$ , we have:

$$(L_X\alpha)(x) = (g\alpha)(x) = (di(X)\alpha)(x).$$

Therefore  $\xi$  is tangent to  $C$  at  $x$ . ■

*Remark.* If  $\xi$  is transversally orientable, there exist contact vector fields  $X$  with empty characteristic hypersurface; there are transverse vector fields to  $\xi$ , i.e. Reeb fields associated to different equations for  $\xi$ .

**Example 2.6** *Any contact vector field  $X$  which is nonsingular or has nondegenerate singularities is generic.*

PROOF Let  $\alpha$  be a local equation for  $\xi$ ; we want to show that  $d(i(X)\alpha)$  is nonzero at every point where  $i(X)\alpha$  is zero. As  $X$  preserves  $\xi$ ,  $L_X\alpha = g\alpha$  for some function  $g$ . Then  $di(X)\alpha = g\alpha - i(X)d\alpha$ .

If  $X$  is nonsingular at  $x \in C$ ,  $(i(X)d\alpha)(x)$  is not proportional to  $\alpha(x)$  since  $d\alpha(x)$  is nondegenerate on  $\xi_x$ . Thus, considering the expression for the Lie derivative,  $di(X)\alpha$  is nonzero at  $x$ .

Now, if  $X$  has at  $x$  a nondegenerate singularity, its linearisation  $A_x : T_x V \rightarrow T_x V$  is invertible. Then the form  $(di(X)\alpha)(x)$ , which is equal to  $\alpha(x) \circ A_x$ , is nonsingular. ■

## C Singularities of contact vector fields

*Remarks.*

- (i) The singularities of a contact vector field lie on its characteristic hypersurface.
- (ii) The divergence of a vector field at a singular point does not depend on the local volume form with which it is calculated: it is the trace of the linearisation.

**Proposition 2.7** *Let  $(V, \xi)$  be a contact manifold of dimension  $2n + 1$  and let  $X$  be a generic contact vector field. To each singularity  $x$  of  $X$  is associated a nonzero real number  $c = c(x)$  (the coefficient of contraction) having the following properties:*

- (i)  $(n + 1)c$  (resp.  $nc$ ) is the divergence of  $X$  at  $x$  (resp. of  $X|_C$  at  $x$ );

(ii) for any local equation  $\alpha$  for  $\xi$ , which induces a form  $\beta$  on  $C$ , we have:

$$(L(X)\alpha)(x) = c\alpha(x) \quad \text{and} \quad (L(X|_C)d\beta)(x) = c d\beta(x).$$

PROOF As  $X$  is contact,  $L(X)\alpha = g\alpha$  for some function  $g$ ; thus, if it exists, the desired coefficient is  $c = g(x)$ . Yet, as  $X$  is generic,  $g(x)$  is nonzero. Then:

$$L(X)d\alpha = dL(X)\alpha = dg \wedge \alpha + g d\alpha. \quad \blacksquare$$

As  $\beta(x) = 0$ , we clearly have:  $(L(X|_C)d\beta)(x) = c d\beta(x)$ . Now, to see that  $c$  does not depend on the choice of  $\alpha$  it suffices to show (i). But as  $\alpha \wedge (d\alpha)^n$  and  $(d\beta)^n$  are local volume forms on  $V$  and  $C$  respectively, (i) follows from the expressions below giving the derivation of a product. For example:

$$\begin{aligned} L(X)(\alpha \wedge (d\alpha)^n) &= (L(X)\alpha) \wedge (d\alpha)^n + \alpha \wedge L(X)(d\alpha)^n \\ &= g\alpha \wedge (d\alpha)^n + n\alpha \wedge L(X)d\alpha \wedge (d\alpha)^{n-1} = (n+1)g\alpha \wedge (d\alpha)^n. \end{aligned}$$

**Corollary 2.8** *If  $x$  is a singularity of a generic contact vector field, the eigenvalue transverse to  $C$  (the tangent space to  $C$  is stable) is equal to  $c$ .*

**Corollary 2.9** *Suppose that  $X$  is, for a certain metric, the gradient of a function  $f$  which has at  $x$  a Morse critical point of index  $i$ . If  $c(x)$  is positive (respectively negative), then  $i$  is at most equal to  $n$  (resp. at least equal to  $n+1$ ).*

PROOF Let  $\alpha$  be an equation for  $\xi$  near  $x$  and  $\beta$  the form induced on  $C$ .

The form  $d\beta(x)$  is a symplectic form on  $T_x C$ . If  $c(x)$  is positive, the tangent space at  $x$  to the stable manifold of  $X|_C$  has dimension  $i$ , since the transverse eigenvalue is positive; moreover it is necessarily isotropic, that is to say contained in its symplectic orthogonal complement (see remark 4.3). Then,  $i$  is at most equal to  $n$ . Similarly, reasoning about the unstable manifold, one sees that, if  $c(x) < 0$ , then  $i \geq n+1$ .  $\blacksquare$

### 3 Convex hypersurfaces

#### A Definition, example

**Definition 3.1** *We say that a hypersurface  $S$  embedded in a contact manifold  $(V, \xi)$  is convex if there exists a contact vector field transverse to  $S$ .*

A convex hypersurface is therefore transversally orientable, that is to say, its tubular neighbourhoods are diffeomorphic to  $S \times \mathbb{R}$ . Also, any germ of a contact vector field along  $S$ , which is transverse to  $S$ , extends to a global contact vector field. The study of convex hypersurfaces is therefore closely linked to that of contact structures on  $S \times \mathbb{R}$  invariant under the vertical vector field  $\partial/\partial t$ , where  $t$  denotes the coordinate on  $\mathbb{R}$ .

**Example 3.2** (*Contactization of an exact symplectic manifold*). We say that a symplectic manifold  $(W, \omega)$  is exact if  $\omega$  is the differential of a 1-form  $\beta$  called the Liouville form. By symplectic duality, it is equivalent to say that there exists on  $W$  a vector field  $X$ , called the Liouville vector field, whose flow dilates  $\omega$  exponentially:  $L(X)\omega = \omega$ . If  $(W, \omega = d\beta)$  is an exact symplectic manifold, the form  $\beta + dt$  defines on  $W \times \mathbb{R}$  a vertically invariant contact structure. Moreover, the Liouville field  $X$ ,  $\omega$ -dual to  $\beta$ , directs the characteristic foliation on the hypersurface  $W \times \{t\}$ ,  $t \in \mathbb{R}$ .

**Remark 3.3** (i) The contact structure thus obtained depends not only on the symplectic structure  $\omega$  but also on the primitive  $\beta$  chosen. We observe however that if we change  $\beta$  to  $\beta + dh$ , where  $h$  is a function on  $W$ , the diffeomorphism  $\phi : W \times \mathbb{R} \rightarrow W \times \mathbb{R}$ , given by  $\phi(x, t) = (x, t + h(x))$ , satisfies  $\phi^*(\beta + dt) = (\beta + dh) + dt$ . This therefore establishes an isomorphism between the two contact structures.

(ii) If  $H \subset W$  is a hypersurface transverse to  $X$ , the form induced by  $\beta$  on  $H$  is contact. In effect,  $\beta \wedge (d\beta)^{n-1} = (1/n) i(X)\omega^n$  induces on  $H$  a volume form.

## B Vertically invariant contact structures

Let  $S$  be a manifold of dimension  $2n$ . A contact structure  $\xi$ , transversally orientable and vertically invariant, on the cylinder  $S \times \mathbb{R}$ , can be defined by a global equation  $\beta + u dt = 0$ , where  $\beta$  and  $u$  are respectively a 1-form and a function on  $S$  such that:

$$\text{the form } \theta = (d\beta)^{n-1} \wedge (u d\beta + n\beta \wedge du) \text{ does not vanish on } S. \quad (1)$$

In fact  $\theta \wedge dt = (\beta + u dt) \wedge (d(\beta + u dt))^n$ .

We observe that:

- (i) The set  $\Sigma$  where  $u = 0$  is the trace on  $S \times \{0\}$  of the characteristic hypersurface of the vector field  $\partial/\partial t$ ; it's a regular hypersurface on which  $\beta$  induces a contact form since, along  $\Sigma$ , 1 says that  $(d\beta)^{n-1} \wedge \beta \wedge du \neq 0$ .
- (ii) On the open set  $\Omega = S \setminus \Sigma$ ,  $\xi$  is still defined by  $\beta/u + dt = 0$  and we have:  $\theta = u^{n+1}(d(\beta/u))^n$ . Thus  $(\Omega \times \mathbb{R}, \xi)$  is the contactization of the exact symplectic manifold  $(\Omega, d(\beta/u))$ .

Let  $Y$  be the tangent vector field to  $S$  defined by:

$$\beta \wedge (d\beta)^{n-1} = i(Y)\theta. \quad (2)$$

This vector field directs the characteristic foliation of  $S \times \{0\}$  (see 1.C) and satisfies the relations below.

- (iii) On  $\Sigma$ :  $Y \cdot u = -1/n$ . In effect:

$$\beta \wedge (d\beta)^{n-1} = -n i(Y)[du \wedge \beta \wedge (d\beta)^{n-1}] = -n(Y \cdot u)\beta \wedge (d\beta)^{n-1},$$

since  $i(Y)[\beta \wedge d\beta^{n-1}] = 0$ .

- (iv) On  $\Omega$ , let  $X$  be the Liouville vector field of  $\beta/u$  defined by  $\beta/u = i(X)d(\beta/u)$ ; we have:

$$X = nuY.$$

In effect:

$$i(X)\theta = u^{n+1}i(X)\left(d\left(\frac{\beta}{u}\right)\right)^n = nu^{n+1}\frac{\beta}{u}\wedge\left(d\left(\frac{\beta}{u}\right)\right)^{n-1} = nu\beta\wedge(d\beta)^{n-1}.$$

**Proposition 3.4** *Let  $S$  be a  $2n$ -dimensional closed manifold and let  $\mathcal{F}$  be a dimension-1 singular foliation on  $S$  (see 1.B). There exists on  $S \times \mathbb{R}$  a vertically invariant contact structure which induces  $\mathcal{F}$  as characteristic foliation on  $S \times \{0\}$  if and only if there exists in  $S$  a hypersurface  $\Sigma$  transverse to  $\mathcal{F}$  (avoiding in particular the singularities of  $\mathcal{F}$ ) such that:*

- (i) *the complement  $S'$  of an open tubular neighbourhood of  $\Sigma$ , of which the fibres are in  $\mathcal{F}$ , is an exact symplectic manifold for which a Liouville field directs  $\mathcal{F}$  and exits transversally on the boundary;*
- (ii) *the involution of the double cover  $\partial S' \rightarrow \Sigma$ , obtained by following the leaves of  $\mathcal{F}$  through the tube, preserves the contact structure induced on  $\partial S'$  (see remark 3.3b but reverses its transverse orientation).*

PROOF We suppose first of all that there exists on  $S \times \mathbb{R}$  a vertically invariant contact structure  $\xi$  which induces  $\mathcal{F}$  as characteristic foliation on  $S \times \{0\}$ . The intersection  $\Sigma$  of  $S$  with the characteristic hypersurface with the vector field  $\partial/\partial t$  is a hypersurface of  $S$  transverse to  $\mathcal{F}$  (see (1) and (3) above). On  $\Omega = S \setminus \Sigma$ , the vertical field is transverse to  $\xi$ , so  $\xi$  is transversally orientable and defined by a unique equation  $\beta + dt = 0$ , where  $\beta$  is necessarily a Liouville form on  $\Omega$ . Using the local equations near  $\Sigma$  and the relations (3) and (4) above, we see that the Liouville field  $X$  associated to  $\beta$  exits along  $\partial S'$ , if  $S'$  is chosen as in the statement. Finally, the contact structure  $\xi'$  defined by  $\beta$  on  $\partial S'$  is the trace on  $\partial S'$  of the contact structure transverse to  $\mathcal{F}$  and invariant under the holonomy of  $\mathcal{F}$ . This implies that the involution of the double cover  $\partial S' \rightarrow \Sigma$  preserves  $\xi'$ ; but, as  $X$  changes the direction of passage of  $\Sigma$ , the transverse orientation of  $\xi'$  is reversed.

Conversely, we now suppose that conditions (i) and (ii) are satisfied. We denote by  $d\beta$  the exact symplectic structure on  $S'$  for which the Liouville field  $X$  directs  $\mathcal{F}$  and exits along  $\partial S'$ .

**Lemma 3.5** *We can suppose, instead of (ii), (ii'): the involution of the double cover  $\partial S' \rightarrow \Sigma$  reverses the form induced by  $\beta$  on  $\partial S'$ .*

PROOF Let  $\bar{S}'$  be the manifold obtained as follows: we reglue to  $S'$  the cylinder  $\partial S' \times [0, \infty)$  along  $\partial S' = \partial S' \times \{0\}$ , connecting  $X$  with the vector field  $\partial/\partial r$  where  $r$  is the coordinate on  $[0, \infty)$ ; we still denote the extended vector field  $X$ . If  $\eta$  denotes the 1-form induced by  $\beta$  on  $\partial S'$ , we extend  $\beta$  to  $\bar{S}'$  by putting

$\beta = e^r \eta$  on  $\partial S' \times [0, \infty)$ . Then  $(\bar{S}', d\beta)$  is an exact symplectic manifold for which the Liouville field is  $X$ .

Now let  $\tau$  be the involution of the double cover  $\partial S' \rightarrow \Sigma$ ; by hypothesis, there exists a function, negative on  $\partial S'$ , denoted  $-e^h$ , satisfying  $\tau^* \eta = -e^h \eta$ ; as  $\tau^2$  is the identity, we have:  $\tau^* h = -h$ . Let  $h_0$  be a minimum of  $h$  on  $\partial S'$  and let

$$S'_0 = S' \cap \{(y, r) \in \partial S' \times [0, \infty) \mid r \leq \frac{1}{2}[h(y) - h_0]\}.$$

Then the form induced by  $\beta$  on  $\partial S'_0 \cong \partial S'$  is:  $\eta_0 = e^{(h-h_0)/2} \eta$ ; then:

$$\tau^* \eta_0 = e^{\tau^*(h-h_0)/2} \tau^* \eta = -e^{-(h+h_0)/2} e^h \eta = -\eta_0.$$

Finally we have an isotopy which sends  $S'_0$  to  $S'$  and respects the foliation by the orbits of  $X$ , which proves the lemma.  $\blacksquare$

On  $S' \times \mathbb{R}$ , the equation  $\beta + dt = 0$  defines a vertically invariant contact structure  $\xi$  which we seek to extend over  $\Sigma \times \mathbb{R}$ . For this, we suppose first of all that  $\Sigma$  is transversally orientable, and we give ourselves a product neighbourhood,  $U \cong \Sigma \times (-1 - \epsilon, 1 + \epsilon)$ , in which the leaves of  $\mathcal{F}$  are the segments  $\{pt\} \times (-1 - \epsilon, 1 + \epsilon)$ . We choose the parametrisation so that:

- $\Sigma \cap U = \Sigma \times \{0\}$  and  $\partial S' \cap U = \Sigma \times \{-1, 1\}$ ;
- on  $S' \cap U$ ,  $X$  has the expression  $-s(\partial/\partial s)$ , where  $s$  is the coordinate on the interval  $(-1 - \epsilon, 1 + \epsilon)$ .

The relations  $L(X)\beta = \beta$ ,  $i(X)\beta = 0$  and the property (ii)' show that  $\beta|_{S' \cap U} = (1/s)\gamma$ , where  $\gamma$  is a contact form on  $\Sigma \times \{1\}$ . Then the form  $\gamma + s dt$  defines on  $U$  a contact structure which coincides with  $\xi$  on  $(U \cap S') \times \mathbb{R}$ .

Finally, if  $\Sigma$  is not transversally orientable, we pass to a cover of  $S$  in which it becomes so and we do the preceding construction in an equivariant manner.  $\blacksquare$

## 4 Convex contact structures

### A Pseudo-gradients of a Morse function

**Definition 4.1** *Let  $f : M \rightarrow \mathbb{R}$  be a Morse function, that is, a function for which all critical points are nondegenerate. We say that a vector field  $X$  is a pseudo-gradient of  $f$  if there exists on  $M$  a Riemannian metric and a positive function  $\rho$  such that, everywhere in  $M$ , we have  $X \cdot f \geq \rho \|df\|^2$ . We then have a similar relation for any other Riemannian metric. For example: the gradient of  $f$  for a given metric verifies this inequality.*

We say that a singularity  $x$  of a vector field  $X$  is *hyperbolic* if the linearisation  $A_x$  of  $X$  at  $x$  is hyperbolic, i.e., not every eigenvalue has zero real part. In this case, the theorem of the stable manifold asserts that the points having  $x$  for  $\omega$ -limit (resp.  $\alpha$ -limit) form an immersed submanifold called the stable manifold (resp. unstable); its tangent space at  $x$  is the stable (resp. unstable) of the linearised field  $A_x$ . It is well known that:

**Proposition 4.2** *Let  $f$  be a Morse function on a manifold  $M$  of dimension  $m$  and let  $X$  be a pseudo-gradient of  $f$ . Then:*

- (i) *the singularities of  $X$  are hyperbolic and are exactly the critical points of  $f$ ;*
- (ii) *at a critical point of  $f$  of index  $i$ , the stable manifold (resp. unstable) of  $X$  has dimension  $i$  (resp.  $m - i$ ).*

**Remark 4.3** *Let  $A$  be a hyperbolic endomorphism of  $\mathbb{R}^{2n}$  and let  $\omega$  be a linear symplectic form on  $\mathbb{R}^{2n}$ . If  $(e^{tA})^*\omega = e^{ct}\omega$  for  $c$  a positive constant and for all reals  $t$ , then the stable manifold  $W^s$  of the linearised field  $A$  is isotropic (i.e. contained in its symplectic orthogonal complement). In effect, for  $v, w \in W^s$ ,  $\omega(v, w) = e^{-ct} \omega(e^{tA}v, e^{tA}w)$  tends towards 0 when  $t$  tends towards  $+\infty$ , then is zero. This allows us to extend corollary 2.9 to the case where  $X$  is a pseudo-gradient of a Morse function.*

## B Notion and condition of convexity for a contact structure

In [EG], Ya. Eliashberg and M. Gromov propose the following definition.

**Definition 4.4** *We say that a contact structure  $\xi$  on a manifold  $V$  is convex if there exists a proper Morse function  $f : V \rightarrow [0, \infty)$  having a complete pseudo-gradient which preserves  $\xi$ .*

The regular levels of  $f$  are then convex hypersurfaces. Moreover it follows from 2.C and 4.A that:

**Proposition 4.5** *Let  $(V, \xi)$  be a contact manifold and  $f : V \rightarrow [0, \infty)$  a proper Morse function. If  $\xi$  is preserved by a pseudo-gradient of  $f$ , the characteristic hypersurface  $C$  of this vector field satisfies the following:*

- (i)  *$f|_C$  is a proper Morse function;*
- (ii) *the critical points of  $f$  are on  $C$  and are exactly the critical points of  $f|_C$ ;*
- (iii) *a critical point of index  $i$  for  $f$  gives, for  $f|_C$ , a critical point of index  $i$  if  $i \leq n$  and of index  $i - 1$  if  $i \geq n + 1$ .*

In part III, we will show how to construct, conversely, convex contact structures on a 3-dimensional manifold  $V$ , starting from a Morse function and a surface in  $V$  satisfying the above conditions.

## C Examples of convex contact structures

**Example 4.6** *(Contactization of a Weinstein manifold)*

**Definition 4.7 (Ya. Eliashberg and M. Gromov, EG)** *We say that a symplectic manifold  $(W, \omega)$  is Weinstein if there exists a proper Morse function  $f_0 : W \rightarrow [0, \infty)$  having a complete pseudo-gradient  $X_0$  which dilates  $\omega$  exponentially:  $L(X_0)\omega = \omega$ . Such a symplectic manifold is therefore exact since, as  $\omega$  is closed, we have  $\omega = d\beta$  where  $\beta = i(X_0)\omega$ .*

In these conditions, the contact structure  $\xi$  defined on  $W \times \mathbb{R}$  by the equation  $\beta + dt = 0$  is convex. In effect, the field  $X = X_0 + t(\partial/\partial t)$  preserves  $\xi$  since  $L(X)(\beta + dt) = \beta + dt$ . Moreover,  $X$  is a complete pseudo-gradient for the proper Morse function  $f : W \times \mathbb{R} \rightarrow [0, \infty)$  given by  $f(x, t) = f_0(x) + t^2$ .

A typical example of a Weinstein manifold is the cotangent space (of any manifold) furnished with its canonical symplectic structure  $\omega$ . In this case, we can choose  $X_0$  so that  $\beta = i(X_0)\omega$  differs from the canonical Liouville form by the differential of a function. The contactization of  $\beta$  is then isomorphic to the canonical contact structure on the space of 1-jets of functions (see remark 3.3a (3.3, (i))): this structure is consequently convex.

**Example 4.8** The contact structure given on  $S^{2n+1}$  by the complex tangencies of the unit sphere in  $\mathbb{C}^{n+1}$  is convex. In effect, if  $z_j = x_j + iy_j$ ,  $1 \leq j \leq n+1$ , are the coordinates, this structure has for example the form induced by  $-\sum y_j dx_j$ ; we check then that the contact vector field associated to the hamiltonian  $x_k$  is a pseudo-gradient of the function  $y_k$ . The characteristic hypersurface on this vector field is the equatorial sphere with equation  $x_k = 0$ .

**Example 4.9** (Canonical structure on the manifold of contact elements.) Let  $\pi : V \rightarrow M$  be the fibre bundle of contact elements on a manifold  $M$  of dimension  $n+1$ . Then the canonical contact structure on  $V$  (see [A]) is convex.

Argument. Given a proper Morse function  $f_0 : M \rightarrow [0, \infty)$ , we choose a complete pseudo-gradient  $X_0$  of  $f_0$  having the following property: at each critical point of  $f_0$ , the eigenvalues of  $X_0$  are distinct and real. Like all vector fields on  $M$ , the vector field  $X_0$  lifts naturally to a contact vector field  $X$  on  $V$ . It turns out that  $X$  is a complete pseudo-gradient for some proper Morse function  $f : V \rightarrow [0, \infty)$ . We obtain  $f$  by perturbing  $f_0 \circ \pi$  as follows: above one such point  $x$ , the vector field  $X$  is tangent to the fibre  $F = \pi^{-1}(x)$  and is none other than the vector field induced naturally by the linearisation of  $X_0$  on the projective cotangent space; as the eigenvalues of  $X_0$  are real and distinct,  $X|_F$  is the gradient of a Morse function  $g : F \rightarrow [0, \infty)$  having exactly  $n+1$  critical points with distinct indices; it is this function  $g$ , properly weighted and extended, that we add to  $f_0 \circ \pi$ .

**Remark 4.10** The characteristic hypersurface of the vector field  $X$  above is the conormal of the vector field  $X_0$ .

## II On the characteristic foliation of surfaces in dimension 3

### 1 Properties of characteristic foliations

We are interested here in singular foliations on a surface  $S$  which can be realised as characteristic foliations by embedding  $S$  in a 3-dimensional contact manifold. In such a manifold, naturally oriented, the normal bundle of  $S$  is isomorphic to

the bundle  $\wedge^2 TS$ ; this allows us to speak of germs of contact structures along  $S$  without specifying the ambient manifold.

## A General form of characteristic foliations

**Definition 1.1** *We say that a singularity  $x$  of a vector field  $Y$  is isochore<sup>1</sup> if the divergence of  $Y$  at  $x$  is zero. An isochore singularity of  $Y$  is then an isochore singularity of  $f \cdot Y$  for any function  $f$ ; this notion is therefore well defined for singular foliations in the sense of 1.1.B.*

**Proposition 1.2** *Let  $\mathcal{F}$  be a singular foliation on a surface  $S$ . We fix an orientation on the manifold  $\wedge^2 TS$  and we are interested only in germs of contact structures along  $S$  which give this orientation.*

- (i)  $\mathcal{F}$  is the characteristic foliation induced on  $S$  by a germ of contact structures if and only if  $\mathcal{F}$  is without isochore singularities.
- (ii) If  $S$  is closed, two germs of contact structures which induce the same characteristic foliation  $\mathcal{F}$  are isomorphic: they are conjugate by a germ of a diffeomorphism which is isotopic to the identity through diffeomorphisms preserving  $\mathcal{F}$ .

**PROOF** (i) The absence of isochore singularities is necessary; in effect, if  $\alpha$  is a contact form which induces on  $S$  a form  $\beta$  null at  $x$ , the form  $d\beta(x)$ , which is none other than  $d\alpha(x)|_{\text{Ker } \alpha(x)}$ , is nondegenerate; in other words, the vector field  $Y$  given near  $x$  by  $\beta = i(Y)d\beta$  has a nonzero divergence at  $x$ .

The converse and (ii) rest on the following fact: let  $S_0$  be an orientable surface; a 1-form  $\alpha = \beta_t + u_t dt$  on  $S_0 \times \mathbb{R}$  is contact if and only if:

$$u_t d\beta_t + \beta_t \wedge \left( du_t - \frac{\partial \beta_t}{\partial t} \right) \text{ is nowhere zero.} \quad (3)$$

In particular,  $\beta_0$  being given, the pairs  $(u_0, (\partial \beta_t / \partial t)|_{t=0})$  which satisfy this inequality for  $t = 0$ , with a fixed sign, form a convex set; or these pairs are those which determine a contact structure. We now suppose that  $S$  is orientable and we take on  $S$  an area form  $\omega$  such that  $\omega \wedge dt$  gives the chosen orientation on  $\wedge^2 TS \cong S \times \mathbb{R}$ . We suppose additionally that  $\mathcal{F}$  is transversally orientable, that is to say, given by an equation  $\beta = 0$ , where  $\beta$  is a 1-form on  $S$ . We denote by  $u$  the function defined on  $S$  by  $d\beta = u\omega$ , and we give ourselves a 1-form  $\gamma$  on  $S$  such that the 2-form  $\beta \wedge \gamma$  is positive or zero relative to  $\omega$ , and strictly positive outside the singular locus of  $\beta$ . We then put  $\beta_t = \beta + t(du - \gamma)$ . The condition 3 shows immediately that the 1-form  $\beta_t + u dt$  defines a contact structure near  $S \times \{0\}$  in  $S \times \mathbb{R}$ ; in effect:

$$u d\beta + \beta \wedge \left( du - \frac{\partial \beta_t}{\partial t} \Big|_{t=0} \right) = u^2 \omega + \beta \wedge \gamma.$$

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<sup>1</sup>Straight from the French!

Yet, as  $\mathcal{F}$  has no isochore singularities,  $u$  is nonzero at each point of the singular locus of  $\beta$ .

Finally, if  $S$  is not orientable or if  $\mathcal{F}$  is not transversally orientable, we compensate for this fault by passing to a cover of order 2 or 4 on which we effect the preceding construction in an invariant manner.

- (ii) By passing eventually to a double cover of  $S$  on which  $\mathcal{F}$  becomes transversally orientable, there remains only the case where the two germs of contact structures are transversally orientable. They then admit equations  $\alpha_0$  and  $\alpha_1$  which induce on  $S$  the same form. The formula 3 shows that the kernel  $\xi_s$  of  $\alpha_s = (1-s)\alpha_0 + s\alpha_1$  is, near  $S$ , a contact structure for all  $s \in [0, 1]$ . We now find, by J. Moser's method, an isotopy  $\varphi_s$ ,  $s \in [0, 1]$ , which takes  $\xi_0$  to  $\xi_s$ , i.e. satisfies:  $\alpha_0 \wedge \varphi_s^* \alpha_s = 0$ . This condition shows that the path  $s \mapsto \varphi_s^* \alpha_s$  lies on the ray  $\{r\alpha_0, r > 0\}$  in the space of 1-forms; in other words:

$$\varphi_s^* \alpha_s \wedge \frac{\partial}{\partial s}(\varphi_s^* \alpha_s) = 0 \quad \text{for all } s.$$

Noting  $X_s$  the infinitesimal generator of  $(\varphi_s)$ , this relation says:

$$(L(X_s)\alpha_s)|_{\xi_s} = -\frac{\partial \alpha_s}{\partial s}|_{\xi_s}.$$

We take for  $X_s$  the vector field verifying at the same time

$$i(X_s)\alpha_s = 0 \quad \text{and} \quad (i(X_s)d\alpha_s)|_{\xi_s} = -\frac{\partial \alpha_s}{\partial s}|_{\xi_s}.$$

This system has a unique solution by definition of contact structure. Moreover, if  $v$  is a vector of  $\xi_x \cap TS = \xi_0 \cap TS$ , we have  $(\partial \alpha_s / \partial s)(v) = 0$ , then  $d\alpha_s(X_s, v) = 0$ . This shows that  $X_s$  is tangent to  $\mathcal{F}$  along  $S$ . We finally use that  $S$  is closed to integrate  $X$ , in an isotopy. ■

## B Generic properties of characteristic foliations

The space of singular foliations on a surface  $S$  (in the sense of I.1.B) has a natural topology as the quotient of the space of plane fields by the null section of  $\wedge^2 TS$ . If now  $S$  is embedded in an oriented 3-dimensional manifold  $V$ , the function which to a plane field on  $V$  associates the induced foliation on  $S$  is open. As the set of contact structures forms an open set, its image is an open set in the space of singular foliations of  $S$ . Also, contact structures being locally stable after a theorem of J. Gray [G], we see:

**Lemma 1.3** *Let  $\mathcal{P}$  be a property of singular foliations which is  $C^\infty$ -generic and let  $S$  be a surface embedded in a contact manifold  $(V, \xi)$ . We can move  $S$  by a  $C^\infty$ -small isotopy so that its characteristic foliation satisfies  $\mathcal{P}$ .*

**Example 1.4** *We say that a vector field on a closed surface is Morse-Smale if it satisfies the three following properties:*

- (i) the singularities and the periodic orbits of  $X$  are hyperbolic;
- (ii) the  $\alpha$ -limit set (resp.  $\omega$ -limit) of every point is a singularity or a limit cycle;
- (iii) there are no connections between saddles.

After a theorem of M. Peixoto, a vector field on a closed orientable surface is  $C^\infty$ -generically Morse-Smale.

Let  $S$  then be a closed orientable surface in a contact manifold  $(V, \xi)$ . If  $\xi$  is transversally orientable, the characteristic foliation of  $S$  is directed by a vector field that we can make Morse-Smale by a  $C^\infty$ -small isotopy of  $S$  in  $V$ .

## 2 Convex surfaces

### A Dividing set of a convex surface

We say that a surface  $S$ , embedded in a 3-dimensional contact manifold  $(V, \xi)$ , is convex if there exists a contact vector field transverse to  $S$ . Such a surface is transversally orientable, therefore orientable. It follows immediately from propositions I.3.4 and II.1.2(b) that:

**Proposition 2.1** *Let  $(V, \xi)$  be a 3-dimensional contact manifold,  $S$  a closed orientable surface embedded in  $V$  and  $\mathcal{F}$  its characteristic foliation. Then the surface  $S$  is convex if and only if there exists on  $S$  a curve  $\Gamma$  transverse to  $\mathcal{F}$ , in general disconnected, which decomposes  $S$  into subsurfaces where  $\mathcal{F}$  can be directed by a dilating vector field, for a certain area form, and exiting through the boundary.*

*Remark.* In particular, if  $S$  is convex, all leaves of  $\mathcal{F}$  cut  $\Gamma$  at most once.

In the following, we will say that  $\Gamma$  is the *dividing set* of  $S$  (see remark 2.3 (2.3)). Giving a contact vector field  $X$  transverse to  $S$  materialises this dividing set as the curve of points of  $S$  where  $X$  is tangent to  $\xi$ .

**Proposition 2.2** (i) *Let  $S$  be a closed surface. Two vertically invariant contact structures on  $S \times \mathbb{R}$  which define the same orientation and induce the same characteristic foliation  $\mathcal{F}$  on  $S \times \{0\}$  are isotopic: they are conjugate by a product diffeomorphism  $\varphi \times Id$ , where  $\varphi$  is isotopic to the identity through diffeomorphisms which preserve  $\mathcal{F}$ . Moreover, if the dividing set of  $S$  associated to the vertical vector field is the same for the two structures, it is preserved all along the isotopy.*

- (ii) *For  $i = 0, 1$  let  $S_i$  be a convex surface in a contact manifold  $(V_i, \xi_i)$ ; let  $\mathcal{F}_i$  be the characteristic foliation and  $X_i$  a contact vector field transverse to  $S_i$  (giving  $X_i$  orients  $S_i$ ). If  $S_0$  and  $S_1$  are closed (compact without boundary) and there exists a diffeomorphism of  $S_0$  in  $S_1$  which respects the orientations and sends  $\mathcal{F}_0$  to  $\mathcal{F}_1$ , then there exists a germ of a contact diffeomorphism, from  $(V_0, S_0)$  to  $(V_1, S_1)$ , which sends  $X_0$  to  $X_1$ .*

PROOF (ii) follows immediately from (i) which is proved exactly like (ii) in proposition 1.2. The isotopy consists of sliding along the leaves of  $\mathcal{F}$  to make one of the contact structures turn into the other. This is only possible in general if  $S$  is closed.  $\blacksquare$

**Remark 2.3** *Proposition 2.2 shows that the characteristic foliation  $\mathcal{F}$  of a convex surface  $S$  totally determines, up to isotopy through curves transverse to  $\mathcal{F}$ , the dividing set  $\Gamma$ , that is to say the trace on  $S$  of the characteristic surface of a transverse contact field. In paragraph 3, we will see in what way this curve reveals the geometry of the characteristic foliation of  $S$ . First we give geometric criteria for convexity and non-convexity and we show in particular that an orientable surface is generically convex. This genericity, exceptional, is related to the fact that every open connected set in  $\mathbb{R}$  (respectively, in  $\mathbb{C}$ ) is convex (respectively pseudo-convex): in dimension 3, the minimal dimension of contact manifolds, convexity is a degenerate property.*

## B Examples of non-convex surfaces

A contact structure on  $S \times \mathbb{R}$  invariant under  $\partial/\partial t$  is (locally) defined by equations of the type  $\beta + u dt = 0$  where  $\beta$  and  $u$  are respectively a 1-form and a function on (a neighbourhood of)  $S$  such that:

$$u d\beta + \beta \wedge du \quad \text{is nowhere zero.} \quad (4)$$

The characteristic foliation  $\mathcal{F}$  of  $S$  is then defined by  $\beta = 0$ . If  $\omega$  is an area form on  $S$  and if  $Y$  is the vector field which directs  $\mathcal{F}$  defined by  $i(Y)\omega = \beta$ , then condition 4 says

$$u \operatorname{div}_\omega(Y) - Y \cdot u \neq 0. \quad (5)$$

This shows immediately that the characteristic foliation of a closed convex surface  $S$  cannot be defined by a closed (nonsingular) form. For example, the invariant tori of the Hopf fibration in  $S^3$  are not convex for the standard structure. We see similarly that, if  $S$  is convex, its characteristic foliation  $\mathcal{F}$  has no closed leaf having a return map tangent to the identity. In effect, in a neighbourhood of one such leaf  $F$ , the foliation  $\mathcal{F}$  admits an equation  $\beta = 0$  where  $d\beta|_F$  is identically zero; it is then impossible to find a function  $u$  such that  $u d\beta + \beta \wedge du$  is nonzero on  $F$  since  $u|_F$  necessarily has critical points. Finally, convexity forbids certain connections of saddles; to be precise, we say:

**Definition 2.4** *Let  $x$  be a non-isochore singularity of a singular foliation  $\mathcal{F}$ . We say that we positively orient  $\mathcal{F}$  at  $x$  when we choose, to direct  $\mathcal{F}$  near  $x$ , a vector field for which the divergence at  $x$  is positive.*

If  $S$  is convex, none of its characteristic foliations joins the stable separatrices of two saddles, each being given their positive orientation.<sup>2</sup> This results for

<sup>2</sup>Unsure: "Si  $S$  est convexe, aucune de ses feuilles caractéristiques ne joint deux selles en étant séparatrice stable de l'une et de l'autre pour leur orientation positive."

example from 5: if, near such a leaf  $F$ , we orient the foliation by a vector field  $Y$  directed from the saddle  $x_0$  towards the saddle  $x_1$ , we must have  $u(x_0)$  negative and  $u(x_1)$  positive. Yet, after 5  $u$  can only be zero when decreasing in the direction of  $Y$ .

## C Examples of convex surfaces

**Definition 2.5** *We say that a singular foliation  $\mathcal{F}$  on a closed surface  $S$  is Morse-Smale if it satisfies the following conditions:*

- (i) *the singularities and the closed leaves of  $\mathcal{F}$  are hyperbolic;*
- (ii) *the limit set of each half-leaf is a singularity or a closed leaf;*
- (iii)  *$\mathcal{F}$  has no connections between saddles.*

*We say that  $\mathcal{F}$  is almost Morse-Smale if it satisfies (i), (ii) and (iii)' when we orient  $\mathcal{F}$  positively near saddles, the associated stable manifolds do not meet each other.*

**Proposition 2.6** *Let  $S$  be an orientable closed surface embedded in a contact manifold  $(V, \xi)$ . If the characteristic foliation  $\mathcal{F}$  of  $S$  is almost Morse-Smale, then  $S$  is convex.*

PROOF After (b) of proposition 1.2 it suffices to construct on  $S \times \mathbb{R}$  a contact structure invariant under  $\partial/\partial t$  which puts  $\mathcal{F}$  as characteristic foliation on  $S \times \{0\}$ . Around each closed leaf (resp. each focus), we take an annulus (resp. a disk) with boundary transverse to  $\mathcal{F}$ . Near saddles, we orient  $\mathcal{F}$  positively. Using (ii) of definition 2.5, we place bands around their stable manifolds so that the union of these annuli, discs and bands form a surface  $S_0$  with boundary transverse to  $\mathcal{F}$  (see Figure 1). By construction after (iii)' of definition 2.5), on a neighbourhood  $U$  of  $S_0$ ,  $\mathcal{F}$  is directed by a field  $Y$  exiting along  $\partial S_0$  for which the singularities have positive divergence. There then exists an area form  $\omega$  on  $S$  such that  $\text{div}_\omega(Y) > 0$  on  $U$ . We put  $u = 1$  then  $u \text{div}_\omega(Y) - Y \cdot u > 0$  on  $U$ .

On the surface with boundary  $S' = \text{adh}(S \setminus S_0)$ ,  $\mathcal{F}$  is a nonsingular foliation transverse to the boundary and without closed leaves.<sup>3</sup> After (ii) as  $S$  is orientable,  $S'$  is a union of annuli foliated by segments going from one boundary to another. We can then conclude by using proposition 2.1 or the following elementary reasoning. We choose on  $S'$  a nonsingular vector field  $Y'$  directing  $\mathcal{F}$  and coinciding with  $\pm Y$  on a collar neighbourhood  $U'$  of  $\partial S'$  in  $U \cap S'$ . We put  $u' = \pm 1$  on  $U'$  accordingly as  $Y' = \pm Y$ ; we then seek to extend to  $S'$  the germ of  $u'$  to the boundary so as to have:  $u' \text{div}_\omega(Y') - Y' \cdot u' > 0$  on  $S'$ . This extension follows immediately from the following remark.

**Remark 2.7** *Let  $h : [0, 1] \rightarrow \mathbb{R}$  be a function positive at 0 and negative at 1. There exists a function  $v : [0, 1] \rightarrow \mathbb{R}$  equal to 1 near 0 and equal to  $-1$  near 1 such that  $vh - dv/d\theta$  is positive; we take  $v(\theta) = w(\theta) \exp(\int_0^\theta h(\sigma) d\sigma)$  where  $w : [0, 1] \rightarrow \mathbb{R}$  is a suitable cutoff function.<sup>4</sup>*

<sup>3</sup>Not sure what adh means. Closure?

<sup>4</sup>“une fonction décroissante convenable.”

### 3 Deformations of characteristic foliations

#### A A reduced form for characteristic foliations

Let  $S$  be a closed orientable surface embedded in a 3-dimensional contact manifold  $(V, \xi)$  with a Morse-Smale characteristic foliation  $\mathcal{F}$ . After proposition 2.6, there exists a germ of a contact vector field transverse to  $S$ . Given any neighbourhood  $U$  of  $S$ , it's easy to extend this germ to a contact vector field for which the flow defines an embedding  $S \times \mathbb{R} \rightarrow V$  with image  $V_0 \subset U$ . On  $V_0 \cong S \times \mathbb{R}$ ,  $\xi_0 = \xi|_{V_0}$  is a contact structure invariant under  $\partial/\partial t$  and the characteristic surface of this contact vector field is a cylinder  $\Gamma \times \mathbb{R}$  where  $\Gamma$  decomposes  $S = S \times \{0\}$  as indicated in 2.1. Then any function  $h : S \rightarrow \mathbb{R}$  has for its graph a convex surface  $S_h$  contained in  $V_0$  having the “same dividing set  $\Gamma$ ” as  $S$ .

**Proposition 3.1** *There exists a function  $h : S \rightarrow \mathbb{R}$  such that the characteristic foliation  $\mathcal{F}_h$  of  $S_h$  is Morse-Smale and gives, on each component  $S'$  of the surface obtained by decomposing  $S_h$  along  $\Gamma$ , the following:*

- (i) *if  $S'$  is a disk,  $\mathcal{F}_h|_{S'}$  has a unique singularity which is a focus and no closed leaves: topologically it is a radial foliation;*
- (ii) *if  $S'$  is not a disk,  $\mathcal{F}_h|_{S'}$  has exactly one closed leaf and only has saddles for singularities.*

Moreover we can choose  $h$  non-positive.

We will show this proposition in C; it follows from proposition 3.6.

#### B Elimination of singularities

**Definition 3.2** *Given a singular foliation without isochore singularities on a surface, we say that a focus  $x_0$  and a saddle  $x_1$  are in simple elimination position (resp. in cyclic elimination position) if when we positively orient the foliation near  $x_1$ , one and only one stable separatrix comes from  $x_0$  (resp. two stable separatrices come from  $x_0$ ).*

**Lemma 3.3 (Elimination lemma)** *(see [El1] theorem 6.1 and [El2]). With the notation and the hypotheses of 3.A, let  $x_0$  and  $x_1$  be a focus and a saddle of  $\mathcal{F}$  in simple or cyclic elimination position.*

- (i) *There exists in  $S$  an annulus  $A$  disjoint from  $\Gamma$  and satisfying:*
  - *the only singularities of  $\mathcal{F}$  on  $A$  are  $x_0$  and  $x_1$ ;*
  - *$\mathcal{F}|_A$  has no closed leaf;*
  - *$\mathcal{F}$  is transverse to the boundary of  $A$ .*

*The two configurations are shown in figures 2 and 3.*

(ii) *There exists a function  $k : A \rightarrow (-\infty, 0]$  with support in the interior of  $A$  and such that the characteristic foliation on the graph of  $k$  has no singularity.*

PROOF (i) Let  $S'$  be the connected component of  $x_1$  in the surface obtained by decomposing  $S$  along  $\Gamma$ . There exists on  $S'$  a vector field which directs  $\mathcal{F}$ , exiting along  $\partial S'$  and dilating some area form on  $S'$ . In particular this vector field positively orients  $\mathcal{F}$  near  $x_1$  and the stable manifold  $W^s(x_1)$  lies in  $S'$ .

If  $x_0$  and  $x_1$  are in cyclic elimination position, we choose for  $A$  an annular neighbourhood of the union  $\{x_0\} \cup W^s$ .

If  $x_0$  and  $x_1$  are in simple elimination position, choose one of the two: either the other branch of  $W^s$  comes from a focus  $x_2$ , or it comes from a closed leaf  $F$  necessarily disjoint from  $\Gamma$ . In the first case, we take for  $A$  a disk neighbourhood of the union  $\{x_0, x_2\} \cup W^s$ , minus a disk around  $x_2$ . In the second case, we first take an annulus  $A'$  around  $F$  with boundary transverse to  $\mathcal{F}$ ; the branch of  $W^s$  which comes from  $F$  then cuts  $\partial A'$  in a point  $x$ ; we take for  $A$  a neighbourhood of the union of the arc which joins  $x$  to  $x_0$  in  $W^s$  and of the component of  $x$  in  $\partial A'$ .

(ii) As  $A$  is disjoint from  $\Gamma$ , the contact structure on  $A \times \mathbb{R}$  is the contactisation of a Liouville form  $\beta$  on  $A$ ; in other words, it has an equation of the form  $\beta + dt = 0$ . Thus, for any function  $k : A \rightarrow \mathbb{R}$  the characteristic foliation on the graph  $A_k$  of  $k$  is defined by  $\beta + dk = 0$ .

Let then  $\omega$  be any area form on  $A$  and  $Y$  the vector field given by  $i(Y)\omega = \beta$ . We seek to add to  $Y$  the  $\omega$ -hamiltonian  $Y_k$  of a function  $k$  with support in the interior of  $A$  such that  $Y + Y_k$  is nonsingular (here  $Y_k$  is defined by  $i(Y_k)\omega = dk$ . For this we take, on  $A$ , a foliation by circles parallel to the boundary which we denote  $\mathcal{G}$ . On a neighbourhood  $B$  of  $\partial A$  in  $A$ ,  $\mathcal{F}$  and  $\mathcal{G}$  are transverse. On  $A \setminus B$ , the vector field  $Y$  is bounded. We therefore choose a function  $k : A \rightarrow (-\infty, 0]$ , with support in the interior of  $A$ , constant on the leaves of  $\mathcal{G}$ , and for which the  $\omega$ -hamiltonian  $Y_k$  is very large on  $A \setminus B$ . Then  $Y + Y_k$  is nonzero on  $A \setminus B$ . On  $B$ ,  $Y$  is nonsingular and is transverse to  $Y_k$  there or  $Y_k$  is nonzero. Then  $Y + Y_k$  is everywhere nonzero. ■

**Remark 3.4** *In the case where  $x_0$  and  $x_1$  are in cyclic elimination position, we thus create a closed leaf.*

*In the case where  $x_0$  and  $x_1$  are in simple elimination position all leaves go to one boundary or the other of the annulus.*

*We can easily check that this construction preserves the Morse-Smale character of the foliation.*

### C End of the proof of proposition 3.1

Let  $S'$  be a component of the surface obtained by decomposing  $S$  along  $\Gamma$ . On  $S'$ , we choose a vector field  $Y$  which directs  $\mathcal{F}$  and which dilates a given area form; the foci and the closed orbits of  $Y$  are then repulsive.

- (i) We suppose that  $S'$  is a disk. Then  $S'$  does not contain any closed orbits since  $Y$  is dilating. If  $x_1 \in S'$  is a saddle of  $Y$ , its stable manifold lies in  $S'$ , therefore  $x_1$  is in elimination position with a focus. When we have eliminated all the saddles, there remains only one focus.
- (ii) We now suppose that  $S'$  is not a disk. The orbits of  $Y$  which leave from a focus  $x_0 \in S'$  cannot go across a closed orbit  $F \subset S'$ . Nor can they all exit since  $S'$  is not a disk. Then, one at least goes towards a saddle  $x_1 \in S'$  such that we can eliminate all the foci of  $S'$ . Now, as the  $\alpha$ -limit set of every point in  $S'$  is in  $S'$ ,  $S'$  contains at least one closed orbit. If it contains only one, we stop. If it contains two,  $F$  and  $F'$ , then  $S'$  is not an annulus and there exists at least one saddle  $x$  in  $S'$  for which one and only one separatrix comes from  $F'$ . In effect, let  $y_1, \dots, y_p$  be the saddles for which one separatrix (and in fact the whole stable manifold) comes from  $F'$ ; the set of points of  $S'$  which have for  $\alpha$ -limit one of the  $y_i$ , or  $F'$ , is a connected component of  $S'$ , but  $S'$  is connected. By the inverse procedure to cyclic elimination, we replace  $F'$  by a focus  $x_0$  and a saddle  $x_1$  in cyclic elimination position. The separatrix of  $x$  which comes from  $F'$  now comes from  $x_0$  so that  $x_0$  and  $x$  are in simple elimination position.

### D Foliations adapted to a given dividing set

Let  $S$  be a convex closed surface in a 3-dimensional contact manifold  $(V, \xi)$ , and let  $X$  be a contact vector field transverse to  $S$  for which the flow defines an embedding  $S \times \mathbb{R} \rightarrow V$ . We denote by  $\Gamma$  the dividing set of  $S$  associated to  $X$ , the curve consisting of points of  $S$  where  $X$  is tangent to  $\xi$ , and we denote by  $S_\Gamma$  the compact surface with boundary obtained by decomposing  $S$  along  $\Gamma$ .

**Definition 3.5** (i) An admissible isotopy of  $S$  in  $V$  is an isotopy of  $S$  through surfaces transverse to  $X$ , which in particular avoid singularities of  $X$ .

(ii) A singular foliation on  $S$  is adapted to  $\Gamma$  if the foliation induced on  $S_\Gamma$  is directed by a vector field which dilates an area form and which exits transversely through the boundary  $\partial S_\Gamma$ .

**Proposition 3.6** Let  $\mathcal{F}$  be a foliation on  $S$  adapted to  $\Gamma$ . Then there exists an admissible isotopy  $\delta_s : S \rightarrow V$ ,  $s \in [0, 1]$ , such that the characteristic foliation on  $\delta_1 S$  is  $\delta_1 \mathcal{F}$ . Moreover, for all  $s \in [0, 1]$ , the dividing set of  $\delta_s S$  associated to  $X$  is  $\delta_s \Gamma$ .

PROOF We denote by  $\mathcal{F}_0$  the characteristic foliation of  $S$  and by  $\xi_0$  the vertically invariant contact structure induced on  $S \times \mathbb{R}$  by the flow of  $X$ ,  $\psi : S \times \mathbb{R} \rightarrow V$ .

We take an area form  $\omega$  on  $S$  such that  $\omega \wedge dt$  orients  $S \times \mathbb{R}$  like  $\xi_0$ ; finally we take a closed tubular neighbourhood  $A$  of  $\Gamma$ , small enough so that  $\mathcal{F}$  and  $\mathcal{F}_0$  foliate it by segments from one boundary to the other.

On  $(S \setminus \text{int}A) \times \mathbb{R}$ ,  $\xi_0$  admits a unique equation of the type  $i(Y_0)\omega + dt = 0$ , where  $Y_0$  is a vector field on  $S \setminus \text{int}A$  which directs  $\mathcal{F}_0$  and which dilates  $\omega$ . Also, as  $\mathcal{F}$  is adapted to  $\Gamma$  there exists on  $S \setminus \text{int}A$  a vector field  $Y$  which directs  $\mathcal{F}$  and which dilates a certain area form; observing that  $\text{div}_{\pm e^g \omega}(Y) = e^{-g} \text{div}_\omega(e^g Y)$ , we replace  $Y$  with a vector field  $Y_1$  which dilates  $\omega$ . For  $s \in [0, 1]$ , we put  $Y_s = (1-s)Y_0 + sY_1$ . Then, the equation  $i(Y_s)\omega + dt = 0$  defines, for each  $s$  in  $[0, 1]$ , a vertically invariant contact structure  $\xi_s$  on  $(S \setminus \text{int}A) \times \mathbb{R}$ . Now, on a small neighbourhood  $U$  of  $A$  in  $S$ , we take vector fields  $Y'_0$  and  $Y'_1$  which respectively direct  $\mathcal{F}_0$  and  $\mathcal{F}$  and which coincide with  $\pm Y_0$  and  $\pm Y_1$  on  $U \cap (S \setminus \text{int}A)$ ; for  $s \in [0, 1]$ , we again set  $Y'_s = (1-s)Y'_0 + sY'_1$ . On  $U \times \mathbb{R}$ , the contact structure  $\xi_0$  is defined by a unique equation of the type  $i(Y'_0)\omega + u_0 dt$ ; the function  $u_0$  is zero on  $\Gamma$ , is equal to  $\pm 1$  wherever  $Y'_0 = \pm Y_0$  and satisfies on  $U$ :  $u_0 \text{div}_\omega(Y'_0) - Y'_0 \cdot u_0 > 0$ .

Then, using remark 2.7, we form a family  $u_s$  of functions on  $U$  such that, for all  $s \in [0, 1]$ , we have  $u_s \text{div}_\omega(Y'_s) - (Y'_s \cdot u_s) > 0$ , with  $u_s = \pm 1$  wherever  $Y'_s = \pm Y_s$ . We thus obtain on  $S \times \mathbb{R}$  a family still denoted  $\xi_s$ ,  $s \in [0, 1]$ , of vertically invariant contact structures; by construction, the characteristic surface of the vertical vector field is  $\Gamma \times \mathbb{R}$  for each structure  $\xi_s$ , and the characteristic foliation induced by  $\xi_1$  on  $S \times \{0\}$  is none other than  $\mathcal{F}$ .

The method of J. Moser (see the proof of proposition 1.2) then provides a family of vertically invariant vector fields on  $S \times \mathbb{R}$  which, since  $S$  is closed, integrate to give an isotopy  $\varphi_s$  satisfying  $\varphi_s^* \xi_s = \xi_0$ ; moreover the diffeomorphisms  $\varphi_s : S \times \mathbb{R} \rightarrow S \times \mathbb{R}$  preserve  $\partial/\partial t$  therefore  $\Gamma \times \mathbb{R}$ ; then  $\varphi_s^{-1}(S \times \{0\})$  is always transverse to  $\partial/\partial t$  and is decomposed by its intersection with  $\Gamma \times \mathbb{R}$ . Composing with a vertical translation, we can arrange that  $\varphi_s^{-1}(X \times \{0\})$  extends to  $S \times (-\infty, 0]$ . We then put  $\delta_s = \psi \circ \varphi_s^{-1}|_{S \times \{0\}}$ . ■

*Remark.* The previous proposition allows us, with lemma 3.3, to eliminate singularities and prove proposition 3.1. Equally it gives other reduced forms for the characteristic foliation of convex surfaces; for example:

**Example 3.7** (*Foliation associated to a handle decomposition*). Let  $(S, X, \Gamma, S_\Gamma)$  be as above. By handle decomposition of  $S_\Gamma$ , we mean a finite collection of arcs  $\gamma_1, \dots, \gamma_r$ , disjoint in  $S_\Gamma$ , going from boundary to boundary, and such that the complement in  $S_\Gamma$  of a regular neighbourhood  $\Omega$  of  $\partial S_\Gamma \cup \gamma_1 \cup \dots \cup \gamma_r$  is a disjoint union of disks  $\Delta_1, \dots, \Delta_q$ .

For any handle decomposition of  $S_\Gamma$ , we associate a singular foliation of  $S_\Gamma$ , unique up to homeomorphism, in the following manner: on each disk  $\Delta_i$ , we put a radial foliation and, on  $\Omega$ , we take the foliation described in Figure 4; this foliation is directed by a vector field exiting through  $\partial S_\Gamma$ , coming in through  $\partial\Omega \setminus \partial S_\Gamma$ , which has no closed orbits and which has for singularities precisely  $r$  saddles with positive divergence for which the unstable manifolds are the  $\gamma_j$ ; note

<sup>5</sup>I think this should be  $Y_0$ . It's  $Y$  in the original.

that the stable manifolds of this saddles come from the centres of the disks  $\Delta_i$ . By regluing, we construct on  $S$  foliations adapted to  $\Gamma$ , without closed leaves.

### III Construction of convex contact structures in dimension 3

#### 1 Convex contact structures and essential surfaces

##### A Existence results

**Definition 1.1** (i) Let  $V$  be a 3-dimensional manifold and  $f : V \rightarrow [0, \infty)$  a proper Morse function. We say that a surface  $C$  embedded in  $V$ , not necessarily connected, is  $f$ -essential if it satisfies the following three properties:

- (a)  $f|_C$  is a proper Morse function;
- (b) all critical points of  $f$  are on  $C$  and are exactly the critical points of  $f|_C$ ;
- (c) a critical point of index 1 or 2 for  $f$  is of index 1 for  $f|_C$ ; equivalently  $f$  and  $f|_C$  have the same local extrema.

(ii) We say that a contact structure on an oriented 3-dimensional manifold is positive if it induces the given orientation.

**Theorem 1.2 (Existence theorem)** Let  $V$  be a 3-dimensional oriented manifold and  $f : V \rightarrow [0, \infty)$  a proper Morse function. There exists on  $V$  a contact structure which is preserved under a complete pseudo-gradient of  $f$  if and only if there exists in  $V$  an  $f$ -essential surface  $C$ .

*Remark.* In I.4, we saw that the existence of an  $f$ -essential surface is necessary; we will prove that this is also sufficient. The problem of existence of essential surfaces for a given function will be discussed in part IV; from this discussion will follow a version of the theorem of R. Lutz and J. Martinet (see [Ma]) for convex contact structures:

**Theorem 1.3** Any oriented 3-dimensional manifold carries a positive convex contact structure.

**Definition 1.4** (Ya. Eliashberg [El3]). We say that a contact structure on a 3-dimensional manifold  $V$  is overtwisted if there exists a 2-dimensional disk embedded in  $V$ , for which the characteristic foliation has a limit cycle (with exactly one singularity in the interior according to [El3], but the arguments of II.3 show that this condition adds nothing).

R. Lutz has described a procedure for constructing on any 3-dimensional manifold an overtwisted contact structure [Lu]; we will give a ‘‘convex’’ version showing that:

**Corollary 1.5** *Any oriented 3-dimensional manifold carries a positive convex overtwisted contact structure.*

After a theorem of M. Gromov and Ya. Eliashberg (see [Gr] and [El1]), overtwisted contact structures are not symplectically fillable (see [El1] and [EG] for the definition). There then exist contact structures which are convex but not symplectically fillable, answering a question of [EG].

## B Scheme of the proof of theorem 1.2

Let  $a_0 < a_1 < \dots$  be the critical values of  $f$ , which we suppose are distinct (only to simplify the exposition), and let  $b_0 < b_1 < \dots$  be intermediate regular values, i.e. so that  $a_0 < b_0 < a_1 < b_1 < \dots$ . We set  $V_k = \{x \in V \mid f(x) \leq b_k\}$  and  $C_k = C \cap V_k$ .

Then  $V_{k+1}$  is obtained from  $V_k$  by attaching a handle of index equal to the index of  $f$  at the critical point  $x_{k+1}$  of value  $a_{k+1}$ . As  $C$  is  $f$ -essential,  $C_{k+1}$  is obtained simultaneously from  $C_k$  by attaching a handle of index equal to the index of  $f|_C$  at  $x_{k+1}$ . Precisely, let  $H_i = D^i \times D^{3-i}$  be a handle of index  $i = 0, 1, 2, 3$ ; the attachment of  $H_i$  to  $V_k$  is given by an embedding  $\varphi : \partial D^i \times D^{3-i} \rightarrow \partial V_k$ ; the pair  $(V_{k+1}, V_k)$  only depends on the isotopy class of  $\varphi$ . For  $j \leq i$ , let  $D^j$  be the sub-disk<sup>6</sup>  $D^j \times \{0\}$  contained in  $D^i$ ; then for an appropriate choice of  $\varphi$ , the handle that we attach to  $C_k$  is  $D^j \times D^{2-j}$  with  $j = 0, 1, 2$  when  $i = 0, 1, 2, 3$ ; we reglue using the restriction of  $\varphi$  to  $\partial D^j \times D^{2-j} \subset \partial D^i \times D^{3-i}$ .

By induction on  $k$ , we construct on  $V_k$  a positive contact structure  $\xi_k$ , thus a pseudo-gradient  $X_k$  for  $f_k = f|_{V_k}$ , which preserves  $\xi_k$  and of which the characteristic surface is  $C_k$ . For this we distinguish four cases corresponding to the different possible indices. It is not necessary to worry about the problem of completeness since we can always adjust it after cutting,<sup>7</sup> in effect:

**Remark 1.6** *Let  $c$  be a given positive number,  $S$  a closed surface and  $\xi$  a vertically invariant contact structure on  $S \times [0, 1]$ . Then there exists on  $S \times [0, 1]$  a contact structure  $\xi'$  having the following properties:*

- (i)  $\xi'$  coincides with  $\xi$  near the boundary;
- (ii)  $\xi'$  is preserved by a vector field  $X'$  which is equal to  $\partial/\partial t$  near the boundary, and for which the orbits are the segments  $\{\cdot\} \times [0, 1]$  covered in a time  $c$ .<sup>8</sup>

**PROOF** We extend  $\xi$  to a vertically invariant contact structure on  $S \times \mathbb{R}$  and we choose a diffeomorphism  $\rho : [0, c] \rightarrow [0, 1]$  which coincides with the identity near 0 and with a translation near  $c$ ; we then take for  $\xi'$  and  $X'$  the images under  $Id \times \rho$  of  $\xi$  and  $\partial/\partial t$ . ■

<sup>6</sup>This seems to be the best translation: "sous-disque".

<sup>7</sup>"Il n'est pas nécessaire de se soucier du problème de complétude car on peut toujours le régler après coup"

<sup>8</sup>Unsure: "les segments  $\{\cdot\} \times [0, 1]$  parcourus en un temps  $c$ ."

## 2 Attachment of handles of index 0 and 3

### A The model

On  $\mathbb{R}^3$  oriented by  $dx \wedge dy \wedge dz$ , the plane field with equation  $dz + uy \, dx + vx \, dy = 0$ ,  $u, v \in \mathbb{R}$ , is a positive contact structure if and only if  $v - u > 0$ . This plane field is preserved by all vector fields of the form

$$ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + cz \frac{\partial}{\partial z}, \quad a, b, c \in \mathbb{R}, \quad \text{with } c = a + b;$$

in effect, their flow at time  $t$  is given by  $(x, y, z) \mapsto (e^{at}x, e^{bt}y, e^{ct}z)$ . Finally, for  $v - u > 0$  and  $c = a + b$ , the characteristic surface of the contact vector field thus defined has equation:  $cz + (au + bv)xy = 0$ .

Let  $\zeta_0$  be the contact structure with equation  $dz - y \, dx + x \, dy = 0$ . The contact vector fields

$$Z_0 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2z \frac{\partial}{\partial z} \quad \text{and } Z_3 = -Z_0$$

both have for characteristic surface the plane  $\{z = 0\}$  are pseudo-gradients respectively of  $g_0 = x^2 + y^2 + z^2$  and  $g_3 = -g_0$ .

We denote by  $H_3$  the handle of index 3:  $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$  which we orient by  $\zeta_0$ ; we denote by  $F_3$  the boundary of  $H_3$  furnished with the induced orientation from the entering vector field  $Z_3$ : this orientation is opposite to the usual orientation of the unit sphere in  $\mathbb{R}^3$  as the boundary of the ball.

### B Handles of index 0

As  $a_0$  is the minimum of  $f$ , there exists a diffeomorphism of  $V_0$  on the closed ball  $B^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$  which respects orientations, which sends  $C_0$  to  $B^3 \cap \{z = 0\}$  and which, up to an affine transformation of  $\mathbb{R}$ , conjugates  $f_0 = f|_{V_0}$  with  $x^2 + y^2 + z^2$ . Then the inverse of this diffeomorphism transforms  $\zeta_0$  into a contact structure  $\xi_0$  on  $V_0$  and sends the vector field  $Z_0$  to a pseudo-gradient  $X_0$  of  $f_0$ ; by construction, this pseudo-gradient preserves  $\xi_0$  and has  $C_0$  as its characteristic surface.

Any ‘‘attachment’’ of a handle of index 0 can be dealt with in the same manner.

### C Handles of index 3

On  $V_k$  we have, by the inductive hypothesis, a contact structure  $\xi_k$ , and also a pseudo-gradient  $X_k$  of  $f_k = f|_{V_k}$  which preserves  $\xi_k$  and has for its characteristic surface  $C_k$ .

**Definition 2.1** *Let  $S \subset \partial V_k$  be a surface. We will say that an isotopy  $\delta_s$  of embeddings of  $S$  in  $V_k$  is admissible if, for all  $s$ ,  $\delta_s S$  is transverse to  $X_k$  in  $V_k$  and cuts  $C_k$  along  $\delta_s(S \cap C_k)$ .*

It is clear that such an isotopy extends to an isotopy of embeddings  $\bar{\delta}_s : V_k \longrightarrow V_k$  which are *admissible* in the following sense:

- for all  $s$ ,  $\bar{\delta}_s$  sends  $C_k$  to  $C_k$ ;
- for all  $s$ ,  $\bar{\delta}_s^*$  is still a pseudo-gradient of  $f_k$  and evidently preserves the positive contact structure  $\bar{\delta}_s^* \xi_k$ .

We now suppose that  $V_{k+1}$  is obtained from  $V_k$  by attaching a handle of index 3, that is, by gluing a ball onto a spherical component  $S$  of  $\partial V_k$ . Simultaneously  $C_{k+1}$  is obtained by attaching to  $C_k$  a disk along  $S \cap \partial C_k$ ; this intersection is then a connected curve  $\Gamma$ . We denote by  $\phi : F_3 \longrightarrow S$  an attaching diffeomorphism which respects orientations and sends  $F_3 \cap \{z = 0\}$  onto  $\Gamma$ .

**Lemma 2.2** *We can find an admissible isotopy  $\delta_s : S \longrightarrow V_k$ ,  $s \in [0, 1]$ , such that there exists a germ of a diffeomorphism  $\psi : (H_3, F_3) \longrightarrow (V_k, \delta_1 S)$  having the following properties:*

- (i)  $\psi|_{F_3}$  is isotopic to  $\delta_1 \phi$  through diffeomorphisms of  $F_3$  in  $\delta_1 S$  which send  $F_3 \cap \{z = 0\}$  to  $\delta_1 \Gamma$ ;
- (ii)  $\psi$  takes  $\zeta_0$  to  $\xi_k$  and  $Z_3$  to  $X_k$ .

PROOF By proposition II.2.2, it suffices to find an admissible isotopy  $\delta_s$  for which the diffeomorphism  $\delta_1 \phi : F_3 \longrightarrow \delta_1 S$  respects orientations and sends the characteristic foliation induced by  $\zeta_0$  to that induced by  $\xi_k$ . This isotopy is immediately given by proposition II.3.6 since the foliation obtained on  $S$  by transporting by  $\phi$  the characteristic foliation on  $F_3$  is adapted to  $\Gamma$ . ■

Now let  $\bar{\delta}_s$  be an admissible isotopy of embeddings  $V_k \longrightarrow V_k$  which extend the isotopy  $\delta_s$  of the above lemma (see 2.1). We can attach  $H_3$  to  $V_k$  by gluing on the one hand  $\bar{\delta}_1^*(\xi_k)$  with  $\zeta_0$ , and on the other  $\bar{\delta}_1^*(X_k)$  with  $Z_3$ . We then extend  $f_k$  to this manifold via a function on  $H_3$  which admits  $Z_3$  as a pseudo-gradient and keeps  $(a_{k+1} - x^2 - y^2 - z^2)$  near the origin.<sup>9</sup> On the other components of  $\partial V_k$ , we add an exterior collar up to the level  $b_{k+1}$ ; there, we extend  $X_k$  trivially then  $\xi_k$  in an invariant manner.

### 3 Attachment of handles of index 1 and 2

#### A The model

On  $\mathbb{R}^3$  oriented by  $dx \wedge dy \wedge dz$ , let  $\zeta_1$  be the positive contact structure with equation  $dz + y dx + 2x dy = 0$ . The contact vector fields

$$Z_1 = 2x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \quad \text{and} \quad Z_2 = -Z_1$$

<sup>9</sup>“une fonction sur  $H_3$  qui admet  $Z_3$  pour pseudo-gradient et vaut  $(a_{k+1} - x^2 - y^2 - z^2)$  près de l'origine.”

have as characteristic surface the plane  $\{z = 0\}$  and are pseudo-gradients respectively of  $g_1 = x^2 - y^2 + z^2$  and  $g_2 = -g_1$ .

Given  $\epsilon > 0$ , we denote by  $H_1 = H_1(\epsilon)$  the handle of index 1  $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 \leq \epsilon^2, y^2 \leq 1\}$  and we denote  $F_1 = F_1(\epsilon)$  the surface  $H_1 \cap \{y = \pm 1\}$ . The giving of  $\zeta_1$  and  $Z_1$  orient  $H_1$  and  $F_1$ . Similarly, we denote by  $H_2$  the handle of index 2  $\{(x, y, z) \in \mathbb{R}^3 \mid y^2 \leq \epsilon^2, x^2 + z^2 \leq 1\}$  and we denote by  $F_2$  the surface  $H_2 \cap \{x^2 + z^2 = 1\}$ ;  $H_2$  and  $F_2$  are oriented by the giving of  $\zeta_1$  and  $Z_2$ . If we parametrise  $F_2$  by  $(\theta, y) \mapsto (x = \sin \theta, y, z = \cos \theta)$ ,  $\theta \in [0, 2\pi]$ , the orientation described previously is given by  $d\theta \wedge dy$ . Otherwise the characteristic foliation induced by  $\zeta_1$  has equation:  $(y \cos \theta - \sin \theta) d\theta + 2 \sin \theta dy = 0$ ; this appears therefore as in figure 5.

We can easily show that:

**Lemma 3.1** *For  $i = 1, 2$  and  $\epsilon > 0$  given, let  $H_i$  be a non-singular germ of a function along  $F_i$ , equal to a negative constant on  $F_i$ . Then  $h_i$  extends to a function on  $H_i$  which coincides with  $g_i$  near the origin, for which  $Z_i$  is a pseudo-gradient.*

## B Handles of index 2

We suppose that  $V_{k+1}$  (resp.  $C_{k+1}$ ) is obtained by attaching to  $V_k$  (resp. to  $C_k$ ) a handle of index 2 (resp. of index 1). This attachment is given by an embedding  $\phi : F_2 \rightarrow S = \partial V_k$  which respects orientations and which meets  $\Gamma = \partial C_k$  exactly along  $F_2 \cap \{z = 0\}$ . The curve of attachment  $\Theta$ , image under  $\phi$  of  $F_2 \cap \{y = 0\}$ , therefore cuts  $\Gamma$  in two points and is in this way divided into two arcs denoted  $\Theta_+$  and  $\Theta_-$ . Finally, we denote by  $S_\Gamma$  the surface obtained by decomposing  $S$  along  $\Gamma$ . To construct the contact structure  $\xi_{k+1}$  and the vector field  $X_{k+1}$  on  $V_{k+1}$  it suffices, by lemma 3.1, to show that:

**Lemma 3.2** *We can find an admissible isotopy  $\delta_s : S \rightarrow V_k$ ,  $s \in [0, 1]$ , such that there exists an annulus  $A$  around  $\Theta$ , and a germ of a diffeomorphism  $\psi : (H_2, F_2) \rightarrow (V_k, \delta_1 A)$  having the following properties:*

- (i)  $\psi|_{F_2}$  is isotopic to  $\delta_1 \phi$  by embeddings of  $F_2$  in  $\delta_1 A$  which meet  $\delta_1 \Gamma$  exactly along  $F_2 \cap \{z = 0\}$ ;
- (ii)  $\psi$  takes  $\zeta_1$  to  $\xi_k$  and  $Z_2$  to  $X_k$ .

**PROOF** We begin by constructing an admissible isotopy  $\delta'_s : S \rightarrow V_k$ ,  $s \in [0, 1]$ , such that there exists an annulus  $A$  around  $\Theta$ , and a diffeomorphism  $\psi' : F_2 \rightarrow \delta'_1 A$  which respects orientations, which meets  $\delta'_1 \Gamma$  exactly along  $F_2 \cap \{z = 0\}$  and which conjugates the induced characteristic foliations respectively by  $\zeta_1$  and  $\xi_k$ . For this, we take two arcs  $\gamma_+$  and  $\gamma_-$  having their endpoints on  $\Gamma$  and satisfying the following conditions (see figure 6):

- $\gamma_+$  and  $\gamma_-$  are contained in a tubular neighbourhood  $\Omega$  of  $\Theta$  in  $S$  and are respectively isotopic to  $\Theta_+$  and  $\Theta_-$  in  $\Omega$ ; moreover this do not cut  $\Gamma$  in their interiors;

- $\gamma_{\pm}$  crosses  $\Theta_{\pm}$  at a single point  $m_{\pm}$ ;
- in  $\Omega$ ,  $\Theta$  cuts  $\Gamma$  between  $\gamma_+$  and  $\gamma_-$ .

We then finish the giving of  $\gamma_+$  and  $\gamma_-$  in a handle decomposition of  $S_{\Gamma}$  (see example II.3.7). The associated foliation induces on  $S$  a foliation  $\mathcal{F}$  adapted to  $\Gamma$  (Definition II.3.5) which, on an annulus  $A$  around  $\Theta$ , is conjugate to the germ of the characteristic foliation of  $F_2$  along the circle  $\{y = 0, x^2 + z^2 = 1\}$ . Proposition II.3.6 gives an admissible isotopy  $\delta'_s : S \rightarrow V_k$  such that  $\delta'_1 A$  has characteristic foliation  $\delta'_1(\mathcal{F})$ . We thus obtain the required diffeomorphism  $\psi' : F_2 \rightarrow \delta'_1 A$ .

Now, we extend  $\psi'$  to a germ of a diffeomorphism, still denoted  $\psi'$ ,  $(H_2, F_2) \rightarrow (V_k, \delta'_1 A)$ , which sends  $Z_2$  to  $X_k$ . In this way,  $\xi_k$  (resp.  $\psi'_* \zeta_1$ ) induces on  $S \times \mathbb{R}$ , via  $\delta'_1$  and the flow of  $X_k$ , a vertically invariant contact structure  $\eta_0$  (resp.  $\eta$ ). It then suffices to establish the following fact:

**Lemma 3.3 (Sub-lemma)** *We can extend  $\eta$  to  $S \times \mathbb{R}$  as a vertically invariant contact structure  $\eta$  giving on  $S \times \{0\}$  the same characteristic foliation and the same dividing set as  $\eta_0$ .*

*Proof that 3.3 implies 3.2.* As  $S$  is closed, we can now argue from the uniqueness of vertically invariant contact structure which induce a given characteristic foliation on  $S \times \{0\}$  (proposition II.2.2): there exists an isotopy  $\varphi_s : S \times \mathbb{R} \rightarrow S \times \mathbb{R}$ , which preserves at once  $\partial/\partial S$  and the levels  $S \times \{t\}$ , such that  $\varphi_1$  straightens  $\eta_1$  to  $\eta_0$ . We then obtain an admissible isotopy  $\delta_s$  and the required diffeomorphism  $\psi$  by correcting by  $\varphi_1$  the isotopy  $\delta'_s$  and the diffeomorphism  $\psi'$ . ■

PROOF (OF 3.3) As  $\psi'$  meets  $\delta'_1 \Gamma$  exactly along  $F_2 \cap \{z = 0\}$ , there exists a function  $h : A \rightarrow \mathbb{R}$  such that, if  $\eta_0$  is defined near a point of  $A \times \mathbb{R}$  by an equation  $\beta + u dt = 0$ , then  $\eta_1$  is defined near this point by  $\beta + e^h u dt = 0$ . We extend  $h$  arbitrarily in a neighbourhood of  $\Gamma$ .

On  $S_{\Gamma} \times \mathbb{R}$ ,  $\eta_0$  induces a vertically invariant contact structure, globally defined by an equation of the form  $i(Y)\omega + u dt = 0$  where:

- $\omega$  is an area form on  $S_{\Gamma}$ ;
- $Y$  is a vector field which exits along  $\partial S_{\Gamma}$  and which directs the foliation on  $S_{\Gamma}$  induced by  $\mathcal{F}$ , that is to say, the foliation associated to the handle decomposition chosen on  $S_{\Gamma}$ ;
- $u$  is a positive function on the interior of  $S_{\Gamma}$ , zero at the boundary and satisfying  $u \div_{\omega}(Y) - Y \cdot u > 0$ .

Let  $A_{\Gamma}$  be the part of  $S_{\Gamma}$  corresponding to  $A$ ; on  $A_{\Gamma} \times \mathbb{R}$ ,  $\eta$  induces a contact structure with equation  $i(Y)\omega + e^h u dt = 0$ . Then:  $u(\div_{\omega}(Y) - Y \cdot h) - Y \cdot u > 0$ , in other words:

$$u(Y \cdot h) < u \div_{\omega}(Y) - Y \cdot u. \quad (6)$$

We must therefore extend  $h$  to  $S_\Gamma$  preserving this inequality. We observe that, on a sufficiently small neighbourhood of  $\partial S_\Gamma$ , the function  $h$  given arbitrarily satisfies 6 since  $u$  is zero on  $\partial S_\Gamma$ . The fact that we can then extend  $h$  results from the two following remarks:

- On the interior of  $S_\Gamma$ , where  $u > 0$  6 says  $Y \cdot h < \div_\omega(Y) - Y \cdot \log u$ . Yet each orbit of  $Y$  which exits  $A_\Gamma$  goes in finite time to  $\partial S_\Gamma$  without cutting  $A_\Gamma$  again. On one such segment of the orbit,  $h$  is given near its endpoints, but the variation of  $-\log u$  is infinite and  $\div_\omega Y$  is bounded; we can therefore extend  $h$  over the segment.
- An orbit of  $Y$  which enters into  $A_\Gamma$  comes to a focus without cutting  $A_\Gamma$  first. On the time interval corresponding to the type  $(-\infty, \tau_0]$ ,  $h$  is only given near  $\tau_0$ . The condition 6, which bounds its derivative above by a strictly positive and decreasing quantity,<sup>10</sup> does not stop us extending  $h$  to a function with compact support. ■

## C Handles of index 1

We suppose that  $V_{k+1}$  (resp.  $C_{k+1}$ ) is obtained from  $V_k$  (resp.  $C_k$ ) by attaching a handle of index 1 at two points  $p$  and  $q$  of  $\Gamma = \partial C_k \subset S = \partial V_k$ . We denote  $p_0$  and  $q_0$  the points with coordinates  $(0, 1, 0)$  and  $(0, -1, 0)$  in  $\mathbb{R}^3$ . By lemma 3.1, it suffices to establish the following fact:

**Lemma 3.4** *There exists a germ of a diffeomorphism  $(V_k, p, q) \longrightarrow (H_1, p_0, q_0)$  which sends  $\xi_k$  to  $\zeta_1$  and  $X_k$  to  $Z_1$ .*

It is in this lemma, for which the proof is easy, that the orientability of  $V$  is needed.

# IV Construction of essential surfaces

In this part, we give methods for constructing, on 3-dimensional manifolds, Morse functions having essential surfaces (see definition 3.1.1). I have had the pleasure of discussing this question with several people, in particular Slava Kharlamov, Francois Laudenbach, Christine Lescop and Alexis Marin; I think them for their suggestions and remarks.

## 1 Some examples

### A Examples of essential surfaces

**Example 1.1** (*F. Laudenbach*). *Let  $V_0$  be a compact 3-dimensional manifold with connected boundary  $C = \partial V_0$ , and let  $f_0 : V_0 \longrightarrow \mathbb{R}$  be a function having the following properties:*

---

<sup>10</sup>Not sure about this: “qui majore sa dérivée par une quantité strictement positive et minorée”

- (i)  $f_0$  is nonsingular and its restriction to  $C$  is a Morse function;
- (ii) any local minimum (resp. local maximum) of  $f_0|_C$  is a local minimum (resp. local maximum) of  $f_0$  on  $V_0$ .

Then there exists on the double  $V = V_0 \cup_C V_0$  a Morse function for which  $C$  is an essential surface.

**Remark 1.2** We will see later (lemma 2.2) that, if  $V_0$  has a function  $f_0$  satisfying (i) and (ii) then  $V_0$  is a handlebody.

PROOF A simple way to construct the double  $V$  of  $V_0$  is the following: we take a Morse function  $g_0 : (V_0, C) \rightarrow ([0, 1], 1)$  without singularities near the boundary. We take on  $V_0 \times [-1, 1]$  the function  $g(x, s) = g_0(x) + s^2$  and we set  $V = \{g = 1\} \subset V_0 \times [-1, 1]$ . It's about a smooth manifold which is identified with the double of  $V_0$  via the two functions  $V_0 \rightarrow V$ ,  $x \mapsto (x, \pm(1 - g_0(x))^{1/2})$ , which sends  $C$  to  $C \times \{0\} \subset V$ .

Now let  $\pi$  be the projection  $V_0 \times [-1, 1] \rightarrow V_0$  and let  $f$  be the restriction to  $V$  of  $f_0 \circ \pi$ . As the kernel of  $d(f_0 \circ \pi) + 2s ds = 0$ , we see that the critical points of  $f$  and all situated on  $C \times \{0\} = V \cap (V_0 \times \{0\})$  and correspond exactly to the critical points of  $f_0|_C$ . Moreover the condition (ii) shows that each local minimum (resp. maximum) of  $f|_C$  is a minimum (resp. maximum) of  $f$ . ■

**Example 1.3** (V.M. Kharlamov). Let  $\Gamma$  be a link in  $S^3$  and  $\pi : V \rightarrow S^3$  a branched double cover over  $\Gamma$ . We suppose there exists a Seifert surface  $C_0$ , bounded by  $\Gamma$ , and a Morse function  $f_0$  on  $S^3$  satisfying the following conditions:

- (i) the critical points of  $f_0$  are in  $C_0 \setminus \Gamma$  and are exactly the critical points of  $f_0|_{C_0}$ ;
- (ii)  $f_0|_{C_0}$  has no local minimum nor local maximum on  $\Gamma$ .

Then  $C = \pi^{-1}(C_0)$  is an essential surface for  $f = f_0 \circ \pi$ .

**Remark 1.4** For several links, we can find a Seifert surface satisfying (i) and (ii) with  $f_0$  the standard height function on  $S^3$ .

PROOF The critical points of  $f$  (resp. of  $f|_C$ ) are of two types:

- the preimages under  $\pi$  of critical points of  $f_0$  (resp. of  $f_0|_{C_0}$ );
- the preimages under  $\pi$  of critical points of  $f_0|_\Gamma$ . For  $f$ , such a point  $x \in V$  is of index 1 or 2 accordingly as  $f_0|_\Gamma$  has at  $\pi(x)$  a minimum or maximum; for  $f|_C$ , such a point is always of index 1, after (ii). ■

## B An example of a function having no essential surface (constructed by C. Lescop)

**Example 1.5** *Let  $p, q$  be relatively prime integers with  $0 \leq q \leq p - 1$ . The oriented lens space  $L(p, q)$  possesses a “canonical” Morse function  $f$  which has exactly one Morse critical point of each index  $0, 1, 2, 3$ . If this function has an essential surface  $C$  then either  $q = 1$ , or  $q = p - 1$ , or  $q$  is odd and  $p = 2(q \pm 1)$ .*

PROOF Let  $b$  be a regular value of  $f$  between critical values of index 1 or 2. We set  $C_0 = C \cap \{f \leq b\}$ ,  $\Gamma = \partial C_0 \subset \{f = b\}$  and we denote by  $\Theta$  the attaching curve of the handle of index 2 on the surface  $\{f = b\}$ , which is an oriented torus. Finally, we denote by  $\mu$  an oriented meridian of this torus ( $\mu$  bounds a disk in  $\{f \leq b\}$ ) and by  $\lambda$  the oriented curve determined by the 2 following conditions:

- the intersection number with  $\mu$  is  $+1$ :  $[\lambda] \cdot [\mu] = +1$ ;
- for a good orientation of  $\Theta$ ,  $[\Theta] = q[\mu] + p[\lambda]$ .

We distinguish two cases accordingly as  $C_0$  is orientable or not.

- (i) If  $C_0$  is orientable, it's an annulus and the curve  $\Gamma$  has two isotopic components,  $\Gamma_0$  and  $\Gamma_1$ , which cut  $\mu$  one time each. Orienting them appropriately, we have, for  $i = 0, 1$ :

$$[\Gamma_i] = m[\mu] + [\lambda],$$

therefore

$$[\Theta] \cdot [\Gamma_i] = pm - q, \quad \text{where } m \in \mathbb{Z}.$$

Thus,  $\Gamma$  cuts  $\Theta$  in at least  $2|pm - q|$  points; yet, since  $C$  exists,  $\Theta$  cuts  $\Gamma$  in exactly two points, so  $pm - q = 0, 1$  or  $-1$ . Then, either  $m = 0$  and  $q = 1$  (unless  $q = 0$  and  $p = 1$ ) or  $m = 1$  and  $q = p - 1$ .

- (ii) If  $C_0$  is not orientable, it's a Mobius strip and  $\Gamma$  is connected. With the appropriate orientation, we have  $[\Gamma] = m[\mu] + 2[\lambda]$ , where  $m$  is an odd integer.

The same argument as above shows that we must have  $mp - 2q = 0, 2$  or  $-2$ ; then,  $m = 1$  and  $p = 2(q \pm 1)$ , with  $q$  odd since  $p$  and  $q$  are relatively prime. ■

**Remark 1.6** *Keeping in mind proposition I.4.5, this example shows that there exist vector fields which, for global reasons, do not preserve any contact structure.*

## 2 A general method for constructing essential surfaces

### A Splitting along an essential surface

**Definition 2.1** *Let  $S$  be a surface and  $\Gamma$  a closed curve on  $S$ , in general disconnected. We will say that  $\Gamma$  divides  $S$  equitably if we can recover  $S$  from two*

subsurfaces, in general non connected, which are both bounded by  $\Gamma$  and have the same Euler characteristic.

**Lemma 2.2** *Let  $V$  be a 3-dimensional manifold,  $f : V \rightarrow [0, \infty)$  a proper Morse function (with distinct critical values), and let  $C$  be an  $f$ -essential surface transversely orientable in  $V$ . Then:*

- (i)  $C$  separates  $V$  into handlebodies;
- (ii)  $C$  cuts each regular level set of  $f$  along a curve which divides it equitably.

We recall that a compact handlebody is a compact 3-dimensional manifold with boundary obtained by attaching to a ball handles of index 1; in the non-compact case, a handlebody is a direct limit<sup>11</sup> of compact handlebodies.

PROOF We choose a transverse orientation on  $C$  and we take two regular values of  $f$ ,  $b_0 < b_1$ , between which  $f$  takes exactly one critical value. For  $i = 0, 1$ , we set  $V_i = \{f \leq b_i\}$ ,  $C_i = C \cap V_i$ ,  $S_i = \{f = b_i\}$  and  $\Gamma_i = C \cap S_i$ . Thus,  $V_1$  (resp.  $C_1$ ) is obtained from  $V_0$  (resp. from  $C_0$ ) by attaching a handle  $H$  (resp.  $K \subset H$ : see the discussion of II.1.B (2.1.2)). We observe that  $K$  separates  $H$  into two components; we denote them  $H^-$  and  $H^+$ ,  $K$  being transversally oriented from  $H^-$  towards  $H^+$ .

If the critical value of  $f$  between  $b_0$  and  $b_1$  is the absolute minimum of  $f$ ,  $C_1$  is a disk which separates the ball  $V_1$  into two balls (with angular boundary)<sup>12</sup>. Moreover,  $\Gamma_1$  is circle and therefore divides the sphere  $S_1$  equitably.

We now suppose that  $V_0$  is the union of two handlebodies, eventually not connected and with angular boundary,<sup>13</sup> which intersect each other exactly along  $C_0$ . We denote them by  $V_0^-$  and  $V_0^+$ ,  $C_0$  being transversally oriented from  $V_0^-$  towards  $V_0^+$ . We suppose additionally that  $\Gamma_0$  divides  $S_0$  equitably.

As the attachment of  $K$  to  $C_0$  must respect the transverse orientation,  $C_1$  separates  $V_1$  into  $V_1^- = V_0^- \cup H^-$  and  $V_1^+ = V_0^+ \cup H^+$ . Thus  $C$  separates  $V$  into two submanifolds  $V^-$  and  $V^+$ .

To show (i) and (ii), we observe that the boundary of  $V_i^\pm$ , for  $i = 0, 1$ , decomposes into two parts:  $C_i$  and  $S_i^\pm = V_i^\pm \cap S_i$ . By hypothesis,  $S_0^-$  and  $S_0^+$  have the same Euler characteristic. Yet:

- If  $H$  has index  $j = 0, 1$ ,  $V_i^\pm$  is obtained from  $V_0^\pm$  by attaching a handle of index  $j$ . Similarly,  $S_i^\pm$  is obtained from  $S_0^\pm$  by attaching a handle of index  $j$ , hence:

$$\chi(S_i^\pm) = (-1)^j + \chi(S_0^\pm),$$

therefore

$$\chi(S_1^+) = \chi(S_1^-).$$

<sup>11</sup>“limite inductive”

<sup>12</sup>Not sure this is right: “à bord anguleux”

<sup>13</sup>Again “à bord anguleux”

- If  $H$  has index  $j = 2, 3$ ,  $V_i^\pm$  is homeomorphic to  $V_0^\pm$ : we simply reglue a ball along a disk contained in the boundary. However,  $S_i^\pm$  is obtained from  $S_0^\pm$  by a “half-surgery” of index  $j$  (it adds a surgery along an arc or a disc lying on the boundary of  $S_0^\pm$ ). We then have:

$$\chi(S_i^\pm) = (-1)^j + \chi(S_0^\pm),$$

therefore, as before,

$$\chi(S_1^+) = \chi(S_1^-). \quad \blacksquare$$

## B The principal construction

**Lemma 2.3** *Let  $S$  be a closed surface,  $\Gamma_0$  a closed curve in  $S$ , not necessarily connected, and  $\alpha$  a simple arc joining in  $S$  two points of  $\Gamma_0$  without other intersection. Then there exists a Morse function  $f : S \times [0, 1] \rightarrow [0, 1]$  satisfying the following conditions:*

- (i)  *$f$  has exactly two critical points with respective indices 1 and 2; moreover, for  $t$  near 0 or 1,  $f|_{(S \times t)} = t$ ;*
- (ii)  *$f$  has an essential surface which cuts  $S \times \{0\}$  along  $\Gamma_0$  and  $S \times \{1\}$  along the curve  $\Gamma_1$ , drawn in figure 7 and obtained as follows: we add a small closed component  $\Gamma'$ , on one side or the other of  $\alpha$  and we do surgery on  $\Gamma_0$  along  $\alpha$  in a neighbourhood of  $\alpha$  avoiding  $\Gamma'$ .*
- (iii) *If  $\Gamma_0$  divides  $S$  equitably,  $C$  is transversely orientable, and  $\Gamma_1$  divides  $S$  equitably also.*

PROOF The method is the following: we realise  $S \times [0, 1]$  by attaching successively a handle of index 1 to  $S \times [0, \epsilon]$ , then a handle of index 2 in elimination position; simultaneously, we attach two handles of index 1 to  $\Gamma_0 \times [0, \epsilon]$  so as to obtain the desired essential surface.

Let  $\alpha_0, \alpha_1$  be elements of  $\Gamma_0$  at the extremities of  $\alpha$ . For  $i = 0, 1$  we choose on  $\alpha_i$  a basis  $(v_i, w_i)$  for the tangent space to  $S$  having the following properties:

- (i)  $v_0$  and  $v_1$  are tangent to  $\Gamma_0$  and on the same side of  $\alpha$ ;
- (ii)  $w_0$  and  $w_1$  are tangent to  $\alpha$  and go back into  $\alpha$ .<sup>14</sup>

We attach then a handle of index 1,  $H_1 = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, y^2 + z^2 \leq 1\}$ , to  $S \times \{\epsilon\}$ , as follows: we send  $(i, 0, 0)$  to  $\alpha_i$ ,  $(\partial/\partial y)(i, 0, 0)$  to  $v_i$  and  $(\partial/\partial z)(i, 0, 0)$  to  $w_i$ . More precisely, the points  $(i, y, 0)$  with  $-1 \leq y \leq 1$  go to  $\Gamma_0$  and the points  $(i, 0, z)$  with  $0 \leq z \leq 1$  go to  $\alpha$ . We thus attach  $K_1 = H_1 \cap \{z = 0\}$  to  $\Gamma_0 \times \{\epsilon\}$ . We denote by  $C_1$  the surface obtained and by  $\Gamma$  its upper<sup>15</sup> boundary:  $\Gamma = \partial C_1 \setminus \Gamma_0$ .

<sup>14</sup>“sont tangents à  $\alpha$  et rentrent dans  $\alpha$ .”

<sup>15</sup>supérieur

For  $i = 0, 1$  we now denote by  $\alpha'_i$  the  $\alpha$  image of  $(i, 0, 1)$  and  $\alpha'$  the sub-arc of  $\alpha$  joining  $\alpha'_0$  and  $\alpha'_1$ . In the lateral boundary of  $H$ ,  $H \cap \{y^2 + z^2 = 1\}$ , we choose an arc  $\alpha''$  transverse to the circles  $\{x = \text{const.}\}$ , isotopic to the fixed extremities of the segment  $\{(x, 0, 1) \mid 0 \leq x \leq 1\}$  and which cut into two points the set  $\{(x, \pm 1, 0) \mid 0 \leq x \leq 1\}$  (see Figure 8).

We then attach a handle of index 2 along  $\Theta = \alpha' \cup \alpha''$ . As  $\alpha''$  is transverse to the circles  $\{x = \text{const.}\}$ , the manifold thus obtained is diffeomorphic to  $S \times [0, 1]$  after the elimination lemma of S. Smale [Mi]. Moreover, by construction,  $\Theta$  cuts  $\Gamma$  in two points in such a way that we can attach (in a unique way) a handle of index 1 to  $C_1$ . We then see painlessly that the surface  $C$  obtained satisfies the required conditions. ■

**Example 2.4** *If  $S$  is the sphere  $S^2$  and if  $\Gamma_0$  is a circle, the curve  $\Gamma_1$  given by lemma 2.2 is formed from three circles linked<sup>16</sup> (i.e. for which the complement is a disjoint union of two disks and two annuli).*

**Corollary 2.5** *There exists a Mores function  $g : S^2 \times [0, 2] \rightarrow [0, 2]$  satisfying the following conditions:*

- (i) for  $t$  near 0 or 2,  $g|_{S^2 \times \{t\}} = t$ ;
- (ii)  $g$  possesses an essential surface  $C$  which cuts  $S^2 \times \{0\}$  and  $S^2 \times \{2\}$  along a circle, and which meets  $S^2 \times \{1\}$  along three linked<sup>17</sup> circles.

PROOF Let  $f : S^2 \times [0, 1] \rightarrow [0, 1]$  be “the” function given by lemma 2.3 taking for  $\Gamma_0$  a circle. We obtain  $g$  by regluing  $f$  with the function:  $S^2 \times [1, 2] \rightarrow [1, 2]$ ,  $(x, y) \mapsto (2 - f(x, 2 - t))$ . ■

**Corollary 2.6** *(Convex version of the Lutz modification [Lu]). Any 3-dimensional manifold which carries a convex contact structure carries a convex overtwisted contact structure.*

**Remark 2.7** *This corollary shows how to deduce corollary III.1.5 from theorem III.1.3.*

PROOF Let  $V$  be a manifold. If there exist on  $V$  a convex contact structure, there exists, by proposition I.4.5, a proper Morse function  $f : V \rightarrow [0, \infty)$  possessing an essential surface  $C$ . For a regular value  $b$  of  $f$ , slightly larger than the absolute minimum and for  $\epsilon$  sufficiently small, the set  $\{b - \epsilon \leq f \leq b + \epsilon\}$  is a product cobordism  $W \cong S^2 \times [0, 1]$  which  $C$  cuts along a cylinder with circular base  $\Gamma \times [0, 1]$ . Corollary 2.5 allows us to replace  $f$  by a proper Morse function  $f' : V \rightarrow [0, \infty)$  with an essential surface  $C'$  which cuts  $S = \{f' = b\} \cong S^2$  along three linked<sup>18</sup> circles. Theorem III.1.2 gives a positive contact structure  $\xi'$  on  $V$  which is invariant under a pseudo-gradient  $X'$  of  $f'$  admitting  $C'$  as a characteristic surface. Proposition II.3.1 shows that then, up to admissible isotopy, the characteristic foliation of  $S$  has two limit cycles, each bounding by a disk with exactly one singularity in its interior. ■

<sup>16</sup>emboîtes

<sup>17</sup>emboîtes

<sup>18</sup>emboîtes

## C An existence theorem

**Theorem 2.8** *On any 3-dimensional manifold, there exists a positive proper Morse function which admits a transversally orientable essential surface.*

**Remark 2.9** *Theorem 2.7, with theorem III.1.2, immediately implies theorem III.1.3.*

PROOF Let  $V$  be a manifold, and  $f : V \rightarrow [0, \infty)$  a proper Morse function, having only one maximum if  $V$  is closed and no maximum if  $V$  is open. Let  $b_0$  and  $b_1$  be two regular values of  $f$  between which  $f$  takes only one critical value  $a$ . We set  $V_i = \{f \leq b_i\}$  for  $i = 0, 1$  and  $S = \{f = b_0\}$ .

If  $a$  is the absolute minimum of  $f$ ,  $f|_{V_1}$  has a transversally orientable essential surface. We therefore suppose now that  $f|_{V_0}$  has a transversally orientable essential surface  $C_0$ , with boundary  $\Gamma_0$ , and we distinguish three cases, according to the index of the critical value  $a$ .

*Index 1.*  $V_1$  is obtained from  $V_0$  by attaching a handle  $H$  of index 1. Changing the attachment of  $H$  by isotopy, we can simultaneously attach along  $C_0$  a handle of index 1 so as to have, for  $f|_{V_1}$ , a transversally orientable essential surface.

*Index 2.* Following lemma 2.2,  $\Gamma_0$  divides  $S$  equitably. Then the attaching curve  $\Theta$  and the handle  $H$  of index  $w$  cuts  $\Gamma_0$  in an even number  $2r$  of points. If  $r = 1$ , we can attach to  $C_0$  a handle of index 1,  $K \subset H$ , which gives for  $f|_{V_1}$  a transversally orientable essential surface. If  $r = 0$ , we move  $\Theta$  by an isotopy to create two intersection points. Now if  $r > 1$ , we apply lemma 2.3 to a sub-arc  $\alpha$  of  $\Theta$  which joins two consecutive intersection points with  $\Gamma_0$ . We thus eliminate the two points replacing  $f|_{V_0}$  with a function  $f'$  which has two critical points with respective indices 1 and 2; we then have a new transversally orientable essential surface  $C'_0$  for which the boundary  $\Gamma'_0$  still divides the surface  $\{f' = b_0\} = \{f = b_0\}$  equitably. Repeating this operation several times, we reduce to the case where  $r = 1$ .

**Remark 2.10** *For compact manifolds with boundary, the proof is finished; for open and non-compact manifolds, we finish with a classic direct limit<sup>19</sup> argument.*

*Index 3.* As  $f$  has a single maximum, the surface  $S = \{f = b_0\}$  is a sphere. If  $\Gamma_0 \subset S$  is a circle, we can, attaching a handle of index 3, reglue a disk to  $C_0$ , which gives the sought transversally orientable essential surface. Now, if  $\Gamma_0$  is not connected, we proceed as follows.

By lemma 2.2,  $\Gamma_0$  divides  $S$  equitably. Then, there exists a component  $\Gamma$  of  $\Gamma_0$  which satisfies the following properties:

- (i)  $\Gamma$  does not bound a disk of  $S \setminus \Gamma_0$ ;
- (ii) in one of the hemispheres of  $S$  bounded by  $\Gamma$ , each component of  $\Gamma_0$  bounds a disk of  $S \setminus \Gamma_0$ ; we denote by  $S'$  this hemisphere and  $S''$  the other.

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<sup>19</sup>limite inductive

(To see that  $\Gamma$  exists, we observe that, if no component of  $\Gamma$  satisfies (i),  $S \setminus \Gamma_0$  is formed on the one hand from a disjoint union of disks, and on the other from a disk with holes.<sup>20</sup> Consequently,  $\Gamma_0$  does not divide  $S$  equitably. To obtain (ii), we choose a component  $\Gamma$  satisfying (i) “minimally”.)

Now, we take a component  $\Gamma'$  of  $\Gamma_0$  in  $S'$  and, in  $S''$ , we choose a component  $\Gamma''$  which we can connect to  $\Gamma$  by an arc  $\alpha^*$  without recutting  $\Gamma_0$ . The inverse construction of that of lemma 2.3 allows us to eliminate all of  $\Gamma'$  by doing connected sum of  $\Gamma$  and  $\Gamma''$  along  $\alpha^*$ . The curve thus obtained again divides  $S$  equitably into two components at least.<sup>21</sup> Repeating this operation, we make  $\Gamma_0$  connected which ends the proof. ■

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<sup>20</sup>disque troué.

<sup>21</sup>a deux composantes de moins.