Floer theoretically essential tori in rational blowdowns

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Joint work with Yankı Lekili, arXiv:1202.5625
to appear in Comm. Math. Helvetici.

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Plan:

1. Symplectic cohomology and geometrically empty manifolds.
2. A_m Milnor fibers and Lagrangians in them.
3. Symplectic topology of the quotients $B_{p,q}$.

Symplectic cohomology.

Symplec cohomology $SH^*(E)$ is an invariant of Liouville manifolds $(E, \omega = d\lambda)$ (e.g. cotangent bundles, affine varieties, or Stein manifolds).

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It can be computed by surgery formula of of Bourgeois-Ekholm-Eliashberg (2012).

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For any Stein manifold E, Abouzaid-Seidel have constructed a diffeomorphic Liouville manifold $\hat{E}$ which has $SH^*(\hat{E}) = 0$ and hence is geometrically empty.

Geometrically empty manifolds with non-vanishing SH.

Theorem (Lekili-M.)

For any $p > q > 1, (p, q) = 1$ there is a Stein surface $B_{p,q}$ which is geometrically empty but has $SH^*(B_{p,q}) \neq 0$ (with any coefficients).

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Remark

Abouzaid-Seidel constructed examples W^{2n} *with* $2n \geq 12$ *such that* $SH^*(W, \mathbb{Z}) \neq 0$*, but* $SH^*(W, \mathbb{Z}_2) = 0$ *so* W *is geometrically empty.*

A_m milnor fibers.

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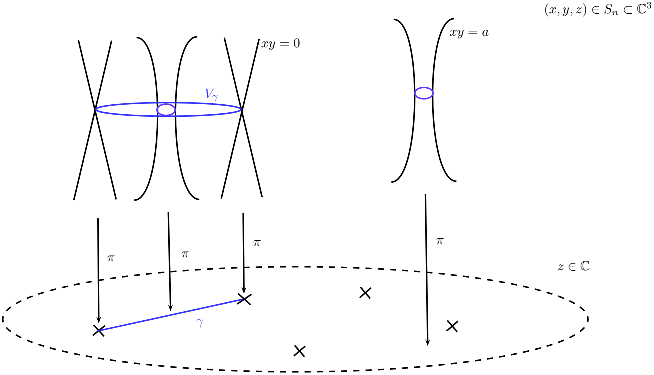
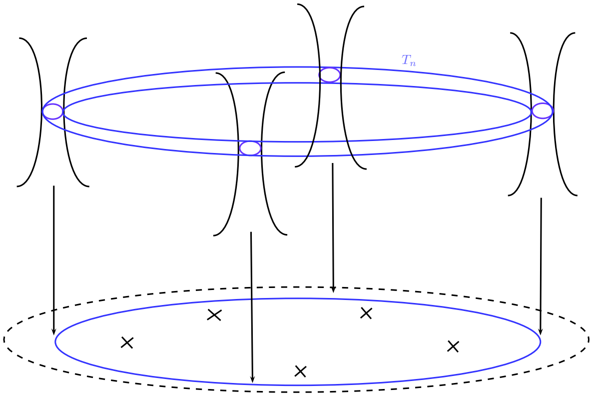
Note $S_0 = \mathbb{C}^2$ and $S_1 = T^*S^2$ with the obvious matching sphere being the zero section S^2.


$(x,y,z) \in S_n \subset \mathbb{C}^3$
$xy = 0$
$xy = a$
V_γ
π
π
π
π
$z \in \mathbb{C}$
γ


T_n

Matching tori in the A_m Milnor fibers.

Proposition

For the matching torus T_n in S_n produced by a closed path enclosing all critical values of π, we have $HF(T_n, \mathbb{Z}_2) = H^*(T_n)$ for $n \geq 1$.

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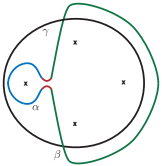
Proof idea:

Holomprhic
curves with boundary on T_n are sections of the Lefschetz fibration π. For $n = 0$ these are given explicitely by Cho or Seidel. We use Seidel's gluing theorem to compute sections in S_n by induction.



Quotients of S_n and their symplectic geometry.

The action.

There are *free* actions of $\mathbb{Z}_p$ on $S_{p-1} = \{(z^p + 2xy = 1\}$ by $\xi\colon (x, y, z) \to (\xi x, \xi^{-1} y, \xi^q z)$, where ξ is the pth root of unity. The quotient is a rational homology ball $B_{p,q}$.

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The essential torus.

T_n projects to $T_{p,q}$, and $HF(T_{p,q}) \neq 0$ by a lifting argument. Hence $T_{p,q}$ is Fler-theoretically essential, and by a theorem of Seidel-Smith $SH^*(B_{p,q}) \neq 0$.

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Theorem (Ritter, 2010)

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Theorem (Khovanov-Seidel, 2002)

For matching spheres V_0 and V_1 over paths γ_0 and γ_1, one has $\operatorname{rk} HF(V_0, V_1) = 2\iota(\gamma_0, \gamma_1)$, *where ι is the minimal geometric intersection number of the two paths.*

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Theorem (Ishii-Ueda-Uehara, 2010)

Any Lagrangian sphere in S_n is equivalent in derived Fukaya category to a matching sphere.

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Any Lagrangian in $B_{p,q}$ would lift to S_{p-1}, and hence is either a sphere or a projective space.

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Spheres are excluded by Euler characteristic as $H_2(B_{p,q}) = 0$. So the only option is a projective space, lifting to some number of spheres and double covered by each. This is impossible for odd p.

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For even $p > 2$ there are at least two spheres in the preimage, taken to each other by deck transfromations of the cover. Replacing these spheres with Fukaya-equivalent matching paths, we get γ and rotated γ which have geometric intersection number 0, which is impossible.

Concluding remarks.

Matching cycles give Lagrangian submanifolds in higher dimensional A_m Milnor fibers with computable Floer theoretic invariants.
In particular, in T^*S^n one gets a large number of Lagrangians of type $S^1 \times S^{k_1} \times \ldots S^{k_s}$ (for $1 + k_1 + \ldots + k_s = n$).

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$B_{p,q}$ carry a (singular) special Lagrangian torus fibration, making them subject to SYZ mirror symmetry conjecture.

Thank you!