MATH 53 SU10, PROBLEM SET 1

1. Find the following integrals:
   (a) \( \int_1^x t^2 e^t \, dt \)
   Solution. Integrate by parts twice (both times “moving” the derivative from the exponential to the polynomial) to obtain
   \[
   \int_1^x t^2 e^t \, dt = \left[ t^2 e^t \right]_1^x - \int_1^x 2te^t \, dt \\
   = x^2 e^x - e - \left( \left[ 2te^t \right]_1^x - \int_1^x 2e^t \, dt \right) \\
   = x^2 e^x - e - 2xe^x + 2e + \left[ e^t \right]_1^x \\
   = x^2 e^x - e - 2xe^x + 2e + 2e^x - 2e \\
   = x^2 e^x - e - 2xe^x + 2e^x. \quad \square
   \]

   (b) \( \int \frac{1}{2 + 2t + t^2} \, dt \)
   Solution. The denominator is an irreducible quadratic so we complete the square: \( 2 + 2t + t^2 = (1 + t)^2 + 1 \). Then
   \[
   \int \frac{1}{2 + 2t + t^2} \, dt = \int \frac{1}{(1 + t)^2 + 1} \, dt \\
   = \arctan(1 + t) + C. \quad \square
   \]

   (c) \( \int \frac{1}{t^2 - 1} \, dt \)
   Solution. By partial fraction decomposition
   \[
   \frac{1}{t^2 - 1} = \frac{1/2}{t - 1} - \frac{1/2}{t + 1} \\
   \]
   so
   \[
   \int \frac{1}{t^2 - 1} \, dt = \int \frac{1/2}{t - 1} - \frac{1/2}{t + 1} \, dt \\
   = \frac{1}{2} \ln(t - 1) - \frac{1}{2} \ln(t + 1) + C. \quad \square
   \]

   (d) \( \int t \sin(t^2) \, dt \)
   Solution. By \( u \)-substitution (with \( u = t^2 \)),
   \[
   \int t \sin(t^2) \, dt = \frac{1}{2} \int 2t \sin(t^2) \, dt = -\frac{1}{2} \cos(t^2) + C. \quad \square
   \]

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(e) \( \int_1^3 \frac{1}{1+t^2} \, dt \)

\textit{Solution.} We compute

\[
\int_1^3 \frac{1}{1+t^2} \, dt = [\arctan t]^3_1 = \arctan 3 - \arctan 1,
\]

which is \( \arctan 3 - \frac{\pi}{4} \). \qed
2. Find all solutions to the following systems of equations:
(a) $2x + y = 7$, $4x + 2y = 14$

   Solution. The two given equations are equivalent (differ by a multiplicative factor of 2) so the solutions of the system coincide with the solutions of either of the two individually. The solutions are parameterized as $x = t$ and $y = 7 - 2t$ for $t$ arbitrary. □

(b) $2x + y = 7$, $4x + 2y = 2$

   Solution. Any solution to the first equation must satisfy (by multiplication by 2) $4x + 2y = 14$ so then the second equation does not hold. Therefore there are no solutions. □

(c) $2x + y = 7$, $4x - 2y = 2$

   Solution. Double the first equation and subtract the second to obtain $4y = 12$ so $y = 3$. Then substituting into either of the original equations implies $x = 2$. The unique solution is $x = 2$ and $y = 3$. □

(d) How many solutions did each of these three equations have? What makes them different from each other?

   Solution. There are infinitely many solutions to (a), no solutions to (b), and exactly one solution to (c). In both (a) and (b), the first and second equations have nonconstant terms ($2x + y$ and $4x + 2y$) that are scalar multiples of one another, but this is not the case for (c). The systems for (a) and (b) differ in the constant terms. □
3. (a) What is the determinant of a 2 by 2 matrix

\[ A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \]

\textit{Solution.} By definition \( \det A = ad - cb \). □

(b) What is the definition of the span of two vectors in \( \mathbb{R}^2 \)?

\textit{Solution.} The span of two vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) is the set of linear combinations of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). In symbols:

\[ \{a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 \mid a_1 \text{ and } a_2 \text{ are scalars}\} \]. □

(c) Is \( \begin{pmatrix} 7 \\ 14 \end{pmatrix} \) in the column space of \( \begin{pmatrix} 2 & 1 \\ 4 & -2 \end{pmatrix} \)? How about \( \begin{pmatrix} 7 \\ 1 \end{pmatrix} \)?

\textit{Solution.} Yes, \( \begin{pmatrix} 7 \\ 14 \end{pmatrix} \) is in the column space of \( \begin{pmatrix} 2 & 1 \\ 4 & -2 \end{pmatrix} \). (You can write the vector as a linear combination of the columns.) No, \( \begin{pmatrix} 7 \\ 1 \end{pmatrix} \) is not in the column space because every element of the column space has the property that twice the first coordinate equals the second, but \( \begin{pmatrix} 7 \\ 1 \end{pmatrix} \) does not enjoy this property. □

(d) If \( A \) is a matrix, what is the definition of its matrix inverse \( A^{-1} \)? Does this inverse always exist? Can a matrix have more than one inverse?

\textit{Solution.} The matrix inverse \( A^{-1} \) of a matrix \( A \) is a matrix such that \( AA^{-1} = I \) and \( A^{-1}A = I \) where \( I \) is the identity matrix of the appropriate size. □

(e) If it exists, what is the inverse of the matrix \( \begin{pmatrix} 2 & 1 \\ 4 & -2 \end{pmatrix} \)?

\textit{Solution.} The inverse does not exist because the determinant is zero (or alternatively because the column space does not contain \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) or \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), which would both necessarily be in the column space in order for there to exist a right inverse to the matrix). □

(f) If it exists, what is the inverse of the matrix \( \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} \)?

\textit{Solution.} In general the inverse of \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) exists if and only if the determinant \( ad - bc \) is nonzero, in which case the inverse is \( \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \). In particular, the inverse of \( \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \) exists and equals \( \frac{1}{4} \begin{pmatrix} -1 & -1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 \\ 1/2 & -1/2 \end{pmatrix} \). □
4. (a) What is the determinant of the matrix

\[ M = \begin{pmatrix} 7 & 2 \\ 2 & 7 \end{pmatrix} \]  

*Solution.* The determinant is \( \det M = 7 \cdot 7 - 2 \cdot 2 = 45 \). □

(b) This matrix \( M \) defines a linear transformation \( T \) of \( \mathbb{R}^2 \).  
   i. What is the geometric shape of the image of the unit square?  
   ii. What is the area of the image of the unit square?  

*Solution.* The geometric shape of the image of the unit square is a parallelogram. Specifically, the parallelogram has vertices the origin, \((7, 2)\), \((2, 7)\), and \((9, 9)\). The transformation \( T \) scales area by a factor of \( \det M = 45 \) and the unit square has area 1 so the area of the image of the unit square is 45. □
5. Draw direction fields for the following ODE:

For (a), draw a direction field by hand using a $3 \times 3$ grid, with $t$-values given by $-1, 0, 1$ and $y$ values given by $0, 1, 2$.

For (a), (b), and (c), use a computer program (see CourseWork for suggestions) to generate a direction field with a large grid (your choice of size, but at least $10 \times 10$), and print it off.

Solution.

\[
y' = 1 - y
\]

\[
y' = t y + 1
\]

\[
y' = (1 - y) y
\]