Do not hand in solutions, but be sure to understand how to do all problems. Some of these will be put on Friday’s quiz.

1. Find the general solutions for each of these three differential equations.
   (a) \( y' = -y + 1 \)
   (b) \( y' = ty + 1 \)
   (c) \( y' = y(1 - y) \)

2. For each of the three differential equations in Problem 1, describe the behavior of solutions as \( t \to +\infty \).
   For each one, choose among the following answers:
   (a) As \( t \to +\infty \), all solutions approach \(+\infty\).
   (b) As \( t \to +\infty \), all solutions approach \(-\infty\).
   (c) As \( t \to +\infty \), all solutions stay bounded.
   (d) As \( t \to +\infty \), some solutions approach \(+\infty\), and some solutions approach \(-\infty\), depending on the initial condition.

   Explain why what you claim is true.

3. A lake, with volume \( V = 100\text{km}^3 \), is fed by a river at the rate of \( r\text{km}^3/\text{yr} \). A sewage pipe dumps contaminated water into the lake at the rate of \( R\text{km}^3/\text{yr} \). By volume, this contaminated water contains 10% contaminants.

   Water leaves from the lake by means of another river. The water flows out of the lake at a rate of \((R + r)\text{km}^3/\text{yr}\) (so the volume of the lake stays constant).

   Let \( x(t) \) be the volume of contaminant in the lake at time \( t \). Let \( c(t) = x(t)/V \) be the concentration of the contaminant.

   (a) Find an initial value problem that describes the concentration of contaminant as a function of time.
   (b) What is the concentration of contaminant after 3 years?

4. (a) Use the Euler method with \( t_0 = 0 \), step-size \( h = 0.5 \) and five steps to find an approximate solution to the initial value problem:

   \[ y' = ty, \quad y(0) = 1. \]

   Graph the resulting curve. (You may use a computer program to generate the graph, but please show the details of the computation of the \( y_k \) you obtain.)

   (b) Solve the initial value problem explicitly, and graph it on the same graph as you used in (a).

5. Solve the initial value problem:

   \[ y' = 2t(\cos(t^2) - y), \quad y(0) = 2. \]

   Is the solution to the initial value problem unique?
6. Find a solution to the following initial value problem:

\[ y' = 2t \cos(t^2)y^2 \quad y(0) = 1. \]

What is its interval of existence (interval of validity)? Is this solution unique?

7. Suppose that the temperature of a cup of coffee changes at a rate proportional to the difference in temperature between it and the surroundings (Newton’s law of cooling).
Suppose the room is at a constant temperature of 60°F. The coffee is initially at 205°F, and in two minutes, has cooled to 190°F.
How long does it take for the coffee to reach 160°F?
Does the initial value problem you introduced have a unique solution?

8. Consider a large rain barrel (shaped like a cylinder) with a small hole near the bottom. Suppose it is initially filled with water to a height \( h_0 \). (Let \( t = 0 \) be this initial time.) Let \( A \) be the cross section of the cylinder. Let \( a \) be the area of the hole.

(a) Torricelli’s law states that the speed of the water flowing out of the hole is proportional to \( \sqrt{2gh} \).
Using this law, make a mathematical model for the height of the water at time \( t \).
(Hint: use the fact that the volume of water contained in the cylinder is \( Ah \).)
(b) Solve the resulting differential equation for the initial value \( h_0 = 1 \) meter. Is this solution unique?
(c) Solve the differential equation for the initial value \( h_0 = 0 \). Is this solution unique? If not, find at least one more solution to the initial value problem.
(d) Comment on the physical meaning of these solutions. (A short comment suffices.)

9. Suppose \( x(t) \) solves the initial value problem:

\[ x' = 2tx - x^2 + e^{2t^2}, \quad x(0) = \frac{1}{2}. \]

Show that \(-e^2 < x(t) < e^2\) for all \( t \). Show that the interval of existence must therefore be the entire real line, i.e. \( x(t) \) must exist for all \( t \).