MATH 147 DIFFERENTIAL TOPOLOGY, SPRING 2008
HOMEWORK 5 SOLUTIONS

Problem 13, page 76 *Consider* S^{k-1} *as a submanifold of* S^k *via the usual embedding mapping* $(x_1, \ldots, x_k) \to (x_1, \ldots, x_k, 0)$. *Show that at* $p \in S^{k-1}$ *the orthogonal complement to* $T_p(S^{k-1})$ *in* $T_p(S^k)$ *is spanned by the vector* $(0, \ldots, 0, 1)$. *Prove that* $N(S^{k-1}; S^k)$ *is diffeomorphic to* $S^{k-1} \times \mathbb{R}$.

Proof. Here we assume $S^k \subset \mathbb{R}^{k+1}$ is defined by $\{x_1^2 + \cdots + x_{k+1}^2 = 1\}$ and S^{k-1} defined by $\{x_1^2 + \cdots + x_k^2 = 1\}$ is embedded in S^k. For any point $p \in S^{k-1}$, T_pS^{k-1} is a subspace with codimension one in T_pS^k. Here we think the tangent space T_pS^{k-1} is the tangent plane in $\mathbb{R}^{k+1}$ and the tangent space T_pS^{k-1} is the tangent plane in $\mathbb{R}^k \simeq \{(x_1, \ldots, x_k, 0) : (x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1}\}$. Since $e_{k+1} \equiv (\underbrace{0, \cdots, 0}_{k \text{ times}}, 1)$ is in T_pS^k but not in $\mathbb{R}^k$. The complement of T_pS^{k-1} in T_pS^k is spanned by e_{k+1}.

To prove that the normal bundle $N(S^{k-1}; S^k)$ is diffeomorphic to $S^{k-1} \times \mathbb{R}$, we consider the map $(p, v) \mapsto (p, v \cdot e_{k+1})$. Then it is easy to check that the map is a diffeomorphism. □

Problem 5, page 83 *Prove that intersection theory is vacuous in contractible manifolds: if* Y *is contractable and* $\dim Y > 0$, *then* $I_2(f, Z) = 0$ *for every* $f : X \to Y$, X *compact and* Z *closed,* $\dim X + \dim Z = \dim Y$.

Proof. First we assume $Z \subsetneqq Y$. Then let $y_0 \in Y \backslash Z$ and define $g : X \to Y$ by $g(x) = y_0$ for all $x \in X$. Since Y is contractible, any two maps from X to Y are homotopic and particularly, f and g are homotopic. Therefore, $I_2(f, Z) = I_2(g, Z) = 0$ since $g^{-1}(Z) = \phi$. □

Problem 8, page 83
(a) *Let* $f : S^1 \to S^1$ *being any smooth map. Prove that there exists a smooth map* $G : \mathbb{R} \to \mathbb{R}$ *such that* $f(\cos y, \sin t) = (\cos g(t), \sin g(t))$, *and satisfying* $g(2\pi) = g(0) + 2\pi q$ *for some integer* g.
(b) *Prove that* $\deg_2(f) = q \mod 2$.

Proof. (a) Let $p : \mathbb{R} \to S^1$ by $p(t) = (\cos t, \sin t)$. It is easy to check that p is a local diffeomorphism. Since f is a continuous function on S^1, we can extend f to be defined on $[0, 2\pi]$ by simply defining $f(2\pi) = f(0)$. We would like to construct g over $[0, 2\pi]$ such that the following diagram commutes and then we are done.

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d, "p"] \\
{[0, 2\pi]} \arrow[ur, "g"] \arrow[r, "f"] & S^1
\end{tikzcd}$$

Let $I \subset [0, 2\pi]$ be the maximal interval so that $g(t)$ is defined continuously and the above diagram commutes. That is

$$I = \Big\{ [0,t] : t = \max s \text{ so that } g : [0,s] \to R \text{ is continuously defined and}$$
$$p \circ g(r) = f(r) \text{ for all } r \in [0,s] \Big\}.$$

First, notice that I is non-empty because we define $g(0) = a$ for some $a \in p^{-1} \circ f(0)$. We will show that I is open and closed, and hence $I = [0, 2\pi]$.

Openness is straightforward from the openness property of a local diffeomorphism. That is, if $g(t)$ is defined, there is an open set V of $g(t)$ so that $V \backsimeq p(V) = f\big((t - \epsilon_1, t + \epsilon_2)\big)$. Therefore, $[0, t + \epsilon_2) \subset I$.

Closedness is from the compactness of S^1. If $\{t_k\} \subset I$ and $t_k \to t_\infty$, then $f(t_k) \to f(t_\infty) \in S^1$ since f is continuous. By the property of local diffeomorphism, there is a neighborhood W of $f(t_\infty)$ so that $p^{-1}(W)$ consists of discrete copies of open sets $\{V_i\}$ in $\mathbb{R}$ and each of them is diffeomorphic to W. By our construction, $g(t_k) \in V_i$ for some i when k large. We then define $g(t_\infty) = p^{-1} \circ f(t_\infty)\big|_{V_i}$. Therefore, I is closed.

We conclude that g is defined on $[0, 2\pi]$ and $g(0) = g(2\pi) + 2\pi q$ for some q because $p \circ g(0) = f(0) = f(2\pi) = p \circ g(2\pi)$. Now we extend g by demanding $g(t + 2\pi) = g(t) + 2\pi q$. To see that g is smooth, we need to consider the case $t = 2\pi k$ for some integer k where we "patch" $g(t)$ together. However, again by the openness of a local diffeomorphism, g is smooth across 0 and across 2π and hence g is smooth for all $2\pi k$ by the way we extend g.

(b) Let $h : S^1 \to S^1$ by $h(t) = tq$, then $\deg_2(h) = q \mod 2$ because

$$I(h, y) = \#\{h^{-1}(y)\} \mod 2 = q \mod 2.$$

Define the homotopy between f and h by $F : S^1 \times [0,1] \to S^1$, $F(t,s) = ((1-s)f(t) + sh)$. Then the homotopy gives

$$\deg_2(f) = \deg_2(h) = q \mod 2.$$

□

Problem11, page 84 *Suppose that $f : X \to Y$ has $\deg_2(f) \neq 0$. Prove that f is onto.*

Proof. If f is not onto, then $Y \setminus f(X)$ is nonempty and open because X is compact. Let $y \in Y \setminus f(X)$. y is a regular value and hence $f(X)$ is trivially transversal to $\{y\}$ and hence $\deg_2(f) = 0 \mod 2$. It contradicts to the assumption that $\deg_2 f \neq 0$. □