MATH 147 DIFFERENTIAL TOPOLOGY, SPRING 2008
HOMEWORK 4 SOLUTIONS

Problem 8, page 55 *Prove that if S^k has a nonvanishing vector field, then its antipodal map is holomorphic to the identity.*

Proof. Without loss of generality, assume $S^k \subset \mathbb{R}^{k+1}$ centered at 0 with radius 1. Let $\vec{v}$ be a nonvanishing vector field and define $F(\vec{x}, t) : S^k \times [0, 1] \to S^k$ to be

$$F(\vec{x}, t) = \cos(\pi t)\vec{x} + \sin(\pi t)\frac{\vec{v}}{|\vec{v}|}.$$

Then it is easy to see that F is a smooth homotopy between the identity map and its antipodal map. □

Problem 9, page 95 *Let $S(X)$ be the set of points $(x, v) \in T(X)$ with $|v| = 1$. Prove that $S(X)$ is a $2k - 1$ dimensional submanifold of $T(X)$; it is called the sphere bundle of X.*

Proof. Define $f : T(X) \to \mathbb{R}$ to be $f(x, v) = |v|^2$. Then for any $(u, w) \in T_{(x,v)}TX$,

$$df(u, w)_{(x,v)} = 2v \cdot w.$$

$df_{(x,v)}$ is surjective if $|v| \neq 0$. In particular 1 is a regular value and therefore, $f^{-1}(1) = S(X)$ is a submanifold with dim $2k - 1$ by the preimage theorem. □

Problem 2, page 62 *Prove that if $f : X \to Y$ is a diffeomorphism of manifolds with boundary, then ∂f maps ∂X diffeomorphically onto ∂Y.*

Proof. We first show that $\partial f : \partial X \to \partial Y$ is well-defined and onto, i.e. $f(\partial X) = \partial Y$.

For any $x \in \partial X$, we prove that $f(x) \in \partial Y$. If not, there is a open neighborhood V of $f(x)$ so that $U = f^{-1}(V)$ is open in X and U, V are diffeomorphic and particularlly $U \setminus \{x\}$ and $V \setminus \{f(x)\}$ are diffeomorphic. However, $U \setminus \{x\}$ and $V \setminus \{f(x)\}$ cannot be diffeomorphic. We get contradiction and hence $f(\partial X) \subset \partial Y$. Similar argument implies for f^{-1} implies $f(\partial X) \supset \partial Y$ and hence $f(\partial X) = \partial Y$.

Define $\partial f : \partial X \to \partial Y$. To see that ∂f is indeed a diffeomorphism, we notice

$$d(\partial f)_z = df_z\Big|_{T_z \partial X}.$$

Since df_z is a diffeomorphism by definition, $df_z\Big|_{T_z \partial X}$ is also and hence ∂f is a diffeomorphism. □

Problem 10, page 64 *Let $x \in \partial X$ be a boundary point. Show that there exists a smooth non-negative function f on some open neighborhood U of x, such that $f(z) = 0$ if and only in $z \in \partial U$, and if $z \in \partial U$, then $df_z(-\vec{n}(z)) > 0$. (Notice that we follow the convention $\vec{n}$ is the unit normal outward vector field on ∂X and we correct the typo in the text.)*

Proof. Let $\pi : \mathbb{H}^k \to \mathbb{R}$, $\pi(x_1, \cdots, x_k) = x_k$, then $\pi \geq 0$. $\pi(p) = 0$ if and only if $p \in \partial\mathbb{H}^k$. Moreover, $d\pi_p(v) > 0$ for any $p \in \partial\mathbb{H}^k$ and for all v pointing inward $\mathbb{H}^k$, i.e. $v = (v_1, \cdots, v_k), v_k > 0$.

Let $\phi : W \to U$ be a local parametrization where W, U are open sets in $\mathbb{H}^k, X$ respectively. Define $f : U \to \mathbb{R}$ by $f = \pi \circ \phi^{-1}$. It is easy to see $f \geq 0$ on U. Moreover, $f(z) = 0$ if and only if $\phi^{-1}(z) \in \partial\mathbb{H}^k$ if and only if $z \in \partial X$. If $z \in \partial U$, $df_z(-\vec{n}(z)) = d\pi_{\phi^{-1}(z)} \circ d\phi_z^{-1}(-\vec{n}(z))$. It is easy to see that $d\phi_z^{-1}(-\vec{n}(z))$ points inward $\mathbb{H}^k$ because for any curve $\gamma(t)$ in X with $\gamma(0) = z$ and $\gamma'(0) = -\vec{n}(z)$,

$$d\phi^{-1}(-\vec{n}(z)) = \lim_{t\to 0} \frac{\phi^{-1}(\gamma(t)) - \phi^{-1}(z)}{t}$$

and we have $\phi^{-1}(\gamma(t)) \in \mathbb{H}^k \setminus \partial\mathbb{H}^k$ and $\phi^{-1}(z) \in \partial\mathbb{H}^k$. Since $d\phi_z^{-1}(-\vec{n}(z))$ points inward $\mathbb{H}^k$, by the property of π, $df_z(-\vec{n}(z)) > 0$.

□