MATH 147 DIFFERENTIAL TOPOLOGY, SPRING 2008
HOMEWORK 3 SOLUTIONS

Problem 4, page 38 *Check that if X is contractible, then all maps of an arbitrary manifold Y into X are homotopic.(And conversely)*

Proof. If X is contractible, there is a homotopy $F : X \times [0,1] \to X$ such that

$$F(\cdot, 0) = id_X, \quad F(x, 1) = x_0 \text{ for some } x_0 \in X.$$

For any maps $f : Y \to X$, let $H : T \times [0,1] \to X$ such that $H(x,t) = F(f(x), t)$. Then

$$H(x, 0) = F(f(x), 0) = f(x), \quad H(x, 1) = F(f(x), 1) = x_0.$$

So f is homotopic to the constant map. Furthermore, homotopy is an equivalence relation, so all smooth maps from Y to X are homotopic to that constant map and hence they are homotopic to one another. □

Problem 6, page 45 *Prove that the sphere S^k is simple connected if $k > 1$.*

Proof. To prove that S^k is simply connected for $k > 1$, we need to prove that for any smooth map $f : S^1 \to S^k$, f is homotopic to a constant map.

We first show that f is not surjective. Notice that $f(x)$ is a critical value for any $x \in S^1$ since $k > 1$ and $d_x f; T_x S^1 \to T_{f(x)} S^k$ cannot be onto. By Sard's theorem, the set of critical values of f has measure zero and therefore, there is p not in $f(S^1)$.

We then use p to define a stereographic projection which is a diffeomorphism between $S^k \setminus \{p\}$ and $\mathbb{R}^k$. W.l.o.g, assume S^k is the unit sphere in $\mathbb{R}^{k+1}$ centered at the origin and p is the north pole $(0, \dots, 0, 1)$. Define $\pi : \mathbb{S}^k \setminus \{p\} \to \mathbb{R}^{k+1}$ by

$$\pi(x_1, \dots, x_{k+1}) = \frac{1}{1 - x_{k+1}}(x_1, \dots, x_k, 0).$$

It is easy to check that $\pi(S^k \setminus \{p\})$ is diffeomorphic to $\mathbb{R}^k$, so $S^k \setminus \{p\}$ is diffeomorphic to $\mathbb{R}^k$.

Notice that $\mathbb{R}^k$ is simply connected because for any $g : S^1 \to \mathbb{R}^k$, define $F : S^1 \times [0,1]$ by $F(x,t) = (1-t)g(x)$, and then F is a homotopy of g and the constant map 0. Now, we consider $g = \pi \circ f : S^1 \to \mathbb{R}^k$ and define $H : S^1 \times [0,1] \to S^k$ by $H(x,t) = \pi^{-1} \circ F(x,t)$, then

$$H(x, 0) = \pi^{-1} \circ g(x) = \pi^{-1} \circ \pi \circ f(x) = f(x), \quad H(x, 1) = \pi^{-1}(0).$$

Hence $f(x)$ is homotopic to a constant map $\pi^{-1}(0)$ and then S^k is simply connected when $k > 1$. □

Problem 13, page 46 *Show that the determinant function on $M(n)$ is Morse if $n = 2$, but not if $n > 2$.*

Proof. For any $A \in M(2)$,

$$A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

and $\det A = x_1x_4 - x_2x_3$.

$$d_A \det = \begin{pmatrix} x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}$$

and A is a critical point if A is the matrix with all entries zero. Moreover,

$$Hess \det \Big|_A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is not singular. Therefore, $\det : M(2) \to \mathbb{R}$ is a Morse function.

For $n > 2$, we will see that the matrix with all entries zero, say A_0, is a degenerate critical point. Let $A = (a_{ij})$ and $S(n)$ be the permutation group, then

$$\det A = \sum_{\sigma \in S(n)} sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

Each entry in $d_A \det$ is a polynomial in a_{ij} of order $n-1$, and therefore A_0 is a critical point. Then each entry in $Hess \det \Big|_{A_0}$ is a polynomial of order $n-2 > 0$ and hence is singular since every entry in A_0 is zero. We conclude that $\det : M(n) \to \mathbb{R}$ is not a Morse function if $n > 2$.

□

Problem 15, page 47 *Let X be a submanifold of $\mathbb{R}^N$. Prove that there exists a linear map $l : \mathbb{R}^N \to \mathbb{R}$ whose restriction to X is a Morse function.*

Proof. By the Theorem on p.43, let $f = 0$ and then $f_a = a_1x_1 + \cdots + a_Nx_N$ is a Morse function for a.e. $a \in \mathbb{R}^N$. Choose the a so that f_a is a Morse function and let $l = f_a$. □