MATH 147 DIFFERENTIAL TOPOLOGY, SPRING 2008
HOMEWORK 2 SOLUTIONS

Problem 1, page 25 *If $f : X \to Y$ is a submersion and U is an open set of X, show that $f(U)$ is open in Y.*

Proof. Assume $\dim X = l. \dim Y = k$ and $l \geq k$. Let ϕ and ψ parametrizations of X and Y respectively. Because $f : X \to Y$ is a submersion, there exists $g(x^1, \dots, x^l) = (x^1, \dots, x^k)$ so that the following diagram commutes (when we restrict some open subsets).

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \, . \\
\mathbb{R}^l \arrow[u, "\phi"] \arrow[r, "g"] & \mathbb{R}^k \arrow[u, "\psi"]
\end{tikzcd}$$

For each point $x \in X$, there exists an open ball $B_{r_x}(x) \subset U$ for radius r_x which may depends on x and $f\big(B_r(x)\big) = \psi \circ g \circ \phi^{-1}\big(B_r(x)\big)$. Since ψ and ϕ^{-1} are diffeomorphisms, they map open sets to open sets. By the definition of g, it also maps open sets to open sets .Therefore we can conclude that $f\big(B_r(x)\big)$ is open and

$$f(U) = f\big(\cup_{x \in U} B_{r_x}(x)\big) = \cup_{x \in U} f\big(B_{r_x}(x)\big)$$

is open. □

Problem 2, page 25 (a) *If X is compact and Y connected, show every submersion $f : X \to Y$ is surjective.*

Proof. By the previous problem, $f(X)$ is open. On the other hand, since f is continuous and X is compact, $f(X)$ is compact and particularly $f(X)$ is closed. Since Y is connected, an open and closed set inside must be either an empty set or Y. Then we conclude $f(X) = Y$, that is, f is surjective.

□

(b) *Show that there exist no submersions of compact manifolds into Euclidean spaces.*

Proof. If there is a submersion $f : X \to \mathbb{R}^n$, by the previous subproblem, $f(X)$ is surjective and hence $f(X) = \mathbb{R}^n$. However $f(X)$ should be compact. We derive a contradiction. □

Problem 11, page 27 (a) *The $n \times n$ matrices with determinant $+1$ form a group denoted $SL(n)$. Prove that $SL(n)$ is a submannifold of $M(n)$ and thus is a Lie group.*

Proof. Let $f : M(n) \to \mathbb{R}$ defined by $f(A) = \det A$ for any $A \in M(n)$. We need to prove that 1 is regular value and then by the Preimage Theorem, $SL(n)$ is a submanifold. In fact, we can show that for any nonzero real number a, a is regular value, that is, we will show that for any A with $\det A \neq 0$,

$$df_A : M(n) \to \mathbb{R}$$

is onto, where we identify $T_A M(n) \approx M(n)$ and $T_{f(A)}\mathbb{R} \approx \mathbb{R}$.

By the hint in the book, for any $b \in \mathbb{R}$, we can consider $B = \frac{b}{n}A$, then it is easy to check that $df_A(B) = b$ as follows.

$$\lim_{t\to 0} \frac{\det(A+tB) - \det A}{t} = \lim_{t\to} \frac{\big((1+t\frac{b}{n})^n - 1\big)\det A}{t} = b.$$

□

(b) *Check that the tangent space to $SL(n)$ at the identity matrix consists of all matrices with trace equal to zero.*

Proof. By the Proposition on p. 24 in the text, we know $T_I SL(n) = \ker df_I$, so we only need to prove

$$\ker df_I = \{A \in M(n) : trA = 0\}.$$

Let $I + tA = (b_{ij})$ and $S(n)$ is the permutation group.

$$\det(I+tA) = \sum_{\sigma\in S(n)} sgn(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma n}.$$

The expression above is a n-th order polynomial in t. The coefficient of t^0 and t^1 are from this term

$$b_{11}b_{22}\cdots b_{nn} = (1+ta_{11})(1+ta_{22})\cdots(1+ta_{nn}) = 1 + t(trA) + O(t^2).$$

Therefore,

$$df_I = \lim_{t\to 0} \frac{\det(I+tA) - \det I}{t} = trA.$$

□

Problem 9, page 33 *Let V be a vector space, and let Δ be the diagonal of $V \times V$. For a linear map $A : V \to V$, consider the graph $W = \{(v, Av) : v \in V\}$. Show that W and Δ are transversal if and only if $+1$ is not an eigenvalue of A .*

Proof. Let $z \in W \cap \Delta$. it is not an empty set since $0 \in W \cap \Delta$.

If W and Δ are transversal, $T_zW + T_z\Delta = T_z(V \times V)$.

$\Longleftrightarrow$ $W + \Delta = V \times V$ because they are linear spaces.

$\Longleftrightarrow$ $W \cap \Delta = \{0\}$ by counting dimension because $\dim W = \dim \Delta = n$ and $\dim V \times V = 2n$.

$\Longleftrightarrow$ There is no non-zero vector v so that $(v, Av) = (v, v)$.

$\Longleftrightarrow$ $+1$ is not an eigenvalue of A.

□