MATH 147 DIFFERENTIAL TOPOLOGY, SPRING 2008
HOMEWORK 1 SOLUTIONS

Problem 8, page 6 *Prove that the hyperboloid in $\mathbb{R}^3$, defined by $x^2 + y^2 - z^2 = a$, is a manifold if $a > 0$. Why doesn't $x^2 + y^2 - z^2 = 0$ define a manifold?*

Proof. Define

$$\phi_1(x,z) = \big(x, \sqrt{z^2 - x^2 + a}, z\big), \phi_2(x,z) = \big(x, -\sqrt{z^2 - x^2 + a}, z)\big),$$

$$\phi_3(y,z) = \big(\sqrt{z^2 - y^2 + a}, y, z\big), \phi_4(x,z) = \big(-\sqrt{z^2 - y^2 + a}, y, z)\big).$$

Denote the hyperboloid to be M. Then

$$\phi_1(x,z) : \{(x,z) \in \mathbb{R}^2 : z^2 - x^2 + a > 0\} \to M \cap \{(x,y,z) : y > 0\},$$

is a diffeomorphism because ϕ_1 is obviously one-one and onto and $d\phi_{(x,z)}$ has rank 2 (here we use the fact that a is not zero).

Similarly,

$$\phi_2(x,z) : \{(x,z) \in \mathbb{R}^2 : z^2 - x^2 + a > 0\} \to M \cap \{(x,y,z) : y < 0\},$$

$$\phi_3(y,z) : \{(y,z) \in \mathbb{R}^2 : z^2 - y^2 + a > 0\} \to M \cap \{(x,y,z) : x > 0\},$$

$$\phi_4(y,z) : \{(y,z) \in \mathbb{R}^2 : z^2 - y^2 + a > 0\} \to M \cap \{(x,y,z) : x < 0\}$$

are diffeomorphisms, so we find local coordinates cover M.

If $a = 0$, then M is not a manifold around the origin $o = (0,0,0)$. Assume it is manifold around $(0,0,0)$ then there is a diffeomorphism $\phi : U \to V$ where U is an open set in $\mathbb{R}^2$ and V is an open set in M containing the origin and say $\phi(x) = o$. If we remove $x \in U$, U is connected. However, $V \setminus o$ is not connected.

□

Problem 9, page 12 (a) *Let $f : X \times Y \to X$ be the projection map $(x,y) \to x$. Show that*

$$df_{(x,y)} : T_x(X) \times T_y(Y) \to T_x(X)$$

is the analogous projection $(v,w) \to v$.

Proof. Let $\phi : U \to X$ and $\psi : V \to Y$ be diffeomorphims onto theirs images where U, V are open sets in Euclidean spaces.

$$\begin{array}{ccc} X \times Y & \xrightarrow{f} & X \\ \uparrow{\scriptstyle \phi\times\psi} & & \uparrow{\scriptstyle \phi} \\ U \times V & \xrightarrow{p} & U \end{array}$$

Then $p(a,b) = (\phi \times \psi) \circ f \circ \phi(a,b) = (a,0)$ is a projection map and $dp_{(a,b)} = I_U$ where I_U is the identity map on U.

$$df_{(x,y)}(v,w) = d\phi \circ dp \circ d\big((\phi \times \psi)^{-1}\big)(v,w)$$

$$= d\phi \circ dp(d\phi^{-1}(v), d\psi^{-1}(w)) = d\phi(d\phi^{-1}(v)) = v.$$

□

(b) *Fixing any $y \in Y$ gives an injection mapping $f : X \to X \times Y$ by $f(x) = (x, y)$. Show that $df_x(v) = (v, 0)$.*

Proof. The argument is similar as in $9(a)$. □

Problem 7, page 18 (a) *Check that $g : \mathbb{R}^1 \to S^1, g(t) = (\cos 2\pi t, \sin 2\pi t)$ is, in fact, a local diffeomorphism.*

Proof.

$$dg_t(v) = (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t)v$$

is an isomorphism from $\mathbb{R}^1$ to S^1 because the rank of dg_t is one for any t. By the inverse function theorem, $g(t)$ is a local diffeomorphism. □

(b) *From Exercise 6, it follows that $G : \mathbb{R}^2 \to S^1 \times S^2, G = g \times g$, is a local diffeomorphism. Also, if L is a line in $\mathbb{R}^2$, the restriction $G : L \times L \to S^1 \times S^1$ is an immersion. Prove that if L has irrational slope, G is one-to-one on L.*

Proof. Let $L = \{(t, s) \in \mathbb{R}^2 : s = at\}$ where a is an irrational number. Since

$$G(t, s) = (\cos 2\pi t, \sin 2\pi t, \cos 2\pi s, \sin 2\pi s).$$

If $G(t_0, at_0) = G(t_1, at_1)$,

$$(\cos 2\pi t_0, \sin 2\pi t_0, \cos 2\pi at_0, \sin 2\pi at_0) = (\cos 2\pi t_1, \sin 2\pi t_1, \cos 2\pi at_1, \sin 2\pi at_1).$$

which implies $t_0 = t_1 +$ some integer and $at_0 = at_1 +$ some integer. Because a is irrational, we have $t_0 = t_1$ and hence G is one-to-one. □

Problem 10, page 19 *(Generalization of the Inverse Function Theorem): Let $f : X \to Y$ be a smooth map that is one-to-one on a compact submanifold Z of X. Suppose that for all $x \in Z$,*

$$df_x : T_x(X) \to T_{f(x)}(Y)$$

is an isomorphism. Then f maps Z diffeomorphically onto $f(Z)$. Prove that f maps an open neighborhood of Z in Z diffeomorphically onto an open neighborhood of $f(Z)$ in Y.

Proof. By the hint in the book, we only need to prove that f is one-to-one on some neighborhood of Z.

Let $U_i = \cup_{z \in Z} B_i(z)$, where $B_i(z)$ is some open neighborhood of z which shrinks to the point z when i tends to infinity. Clearly $Z \subset U_i$ and U_i is open. If f is not one-to-one on some neighborhood of Z, we can find sequences of points $\{a_i\}, \{b_i\} \subset U_i$ such that $a_i \neq b_i$ and $f(a_i) = f(b_i)$. Because $\{a_i\}, \{b_i\}$ are bounded sequences, there exist convergent subsequences and w.o.l.g we assume they are still indexed the same, say $a_i \to a$ and $b_i \to b$. We know $a, b \in Z$ because a_i, b_i can be arbitrarily close to Z. Then by the assumption that f on Z is one-to-one, we get $a = b \equiv z$. However, it contradicts that the non-singularity of df_z. Because df_z is not singular, by the inverse function theorem (for a point), f is a local diffeomorphism around a neighborhood of z.

□