MATH 110: LINEAR ALGEBRA
SPRING 2007/08
PROBLEM SET 9

1. A matrix $S \in \mathbb{R}^{n\times n}$ is called *skew symmetric* if $S^\top = -S$.

(a) For any matrix $A \in \mathbb{R}^{n\times n}$ for which $I + A$ is nonsingular, show that

$$(I - A)(I + A)^{-1} = (I + A)^{-1}(I - A). \tag{1.1}$$

We will write

$$\frac{I - A}{I + A}$$

for the matrix in (1.1). [Note: In general, $AB^{-1} \neq B^{-1}A$ and so

$$\frac{A}{B}$$

is ambiguous since it could mean either AB^{-1} or $B^{-1}A$.]

(b) Let $Q \in \mathbb{R}^{n\times n}$ be an orthorgonal matrix such that $I + Q$ is nonsingular. Show that

$$\frac{I - Q}{I + Q}$$

is a skew symmetric matrix.

(c) Let $S \in \mathbb{R}^{n\times n}$ be a skew symmetric matrix. Show that

$$\frac{I - S}{I + S}$$

is an orthogonal matrix.

(d) Why is it unnecessary to require that $I + S$ be nonsingular in (c)? [Hint: Problem **3** below.]

2. Let $A, B \in \mathbb{R}^{n\times n}$. Let $\lambda_a \in \mathbb{R}$ be an eigenvalue of A and $\lambda_b \in \mathbb{R}$ be an eigenvalue of B.

(a) Is it always true that $\lambda_a\lambda_b$ is an eigenvalue of AB? Is it always true that $\lambda_a + \lambda_b$ is an eigenvalue of $A + B$?

(b) Show that $\lambda \in \mathbb{R}$ is an eigenvalue of AB iff $\lambda \in \mathbb{R}$ is an eigenvalue of BA. [Hint: Homework **8**, Problem **1**(a).]

(c) Let $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{R}$. Show that

$$\alpha_0 + \alpha_1\lambda_a + \alpha_2\lambda_a^2 + \cdots + \alpha_d\lambda_a^d \in \mathbb{R}$$

is an eigenvalue of the matrix

$$\alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_d A^d \in \mathbb{R}^{n\times n}.$$

(d) Show that if A is nonsingular, then $\lambda_a \neq 0$ and $1/\lambda_a$ is an eigenvalue of A^{-1}.

3. A matrix $M \in \mathbb{R}^{n\times n}$ is called *positive semidefinite* if

$$\mathbf{x}^\top M\mathbf{x} \geq 0$$

for all $\mathbf{x} \in \mathbb{R}^n$. M is called *positive definite* if (i) M is positive semidefinite; and (ii) $\mathbf{x}^\top M\mathbf{x} = 0$ only if $\mathbf{x} = \mathbf{0}$.

(a) Show that every positive definite matrix is nonsingular (ie. invertible).

(b) Show that if M is positive semidefinite and $\lambda \in \mathbb{R}$ is an eigenvalue of M, then $\lambda \geq 0$.

(c) Show that if M is positive definite and $\lambda \in \mathbb{R}$ is an eigenvalue of M, then $\lambda > 0$.

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(d) Let M be positive definite and let

$$S_+ := \frac{1}{2}(M + M^\top) \quad \text{and} \quad S_- := \frac{1}{2}(M - M^\top).$$

Show that S_+ is a symmetric positive definite matrix and that

$$\mathbf{x}^\top M\mathbf{x} = \mathbf{x}^\top S_+\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$. Show that S_- is a skew-symmetric matrix, and that

$$\mathbf{x}^\top S_-\mathbf{x} = 0$$

for all $\mathbf{x} \in \mathbb{R}^n$.

(e) Show that if M is positive definite, then $\langle \cdot\,, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top M\mathbf{y},$$

is an inner product on $\mathbb{R}^n$.

4. Let $A \in \mathbb{R}^{m\times n}$.

(a) Show that $A^\top A$ and $AA^\top$ are symmetric positive semidefinite matrices. Hence deduce that singular values are always nonnegative.

(b) Show that if A is *full rank*, ie. $\operatorname{rank}(A) = \min\{m, n\}$, then either $A^\top A$ or $AA^\top$ must be positive definite.

(c) Let $\lambda \in \mathbb{R}$ be an eigenvalue and $\mathbf{x} \in \mathbb{R}^{m+n}$ be a corresponding eigenvector of the matrix

$$\begin{bmatrix} O & A \\ A^\top & O \end{bmatrix} \in \mathbb{R}^{(m+n)\times(m+n)},$$

(written in block matrix form where O denotes a zero matrix of the appropriate size). Show that $\sigma = |\lambda|$ is a singular value of A and if $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$ are such that

$$\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix},$$

then $\mathbf{u}$ is a left singular vector and $\mathbf{v}$ is a right singular vector of A corresponding to the singular value σ.

(d) Let $m = n$, ie. A is a square matrix. Show that

$$\frac{1}{2}\mathbf{x}^\top(A + A^\top)\mathbf{x} \le (\mathbf{x}^\top A^\top A\mathbf{x})^{\frac{1}{2}}$$

for all $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\|_2 = 1$.