MATH 110: LINEAR ALGEBRA
SPRING 2007/08
PROBLEM SET 5

If $\mathcal{T} : V \to V$ is a linear operator, we will write $\mathcal{T}^n = \overbrace{\mathcal{T} \circ \cdots \circ \mathcal{T}}^{n \text{ times}}$, ie. $\mathcal{T}$ composed with itself n times. For example, $\mathcal{T}^2 = \mathcal{T} \circ \mathcal{T}, \mathcal{T}^3 = \mathcal{T} \circ \mathcal{T} \circ \mathcal{T}$, etc. We will let $\mathcal{O} : V \to W$ denote the zero linear transformation, ie. $\mathcal{O}(\mathbf{v}) = \mathbf{0}_W$ for all $\mathbf{v} \in V$. The set of all linear transformations from V to W is denoted $\operatorname{Hom}_{\mathbb{F}}(V, W)$. We write $\operatorname{End}_{\mathbb{F}}(V)$ for $\operatorname{Hom}_{\mathbb{F}}(V, V)$.

1. Let U, W, V be finite-dimensional vector spaces over $\mathbb{F}$. Let $\alpha, \beta \in \mathbb{F}$.

(a) Let $\mathcal{T} : V \to W$ be a linear transformation. Show that
$$\operatorname{rank}(\mathcal{T}) \leq \dim(V).$$

(b) Let $\mathcal{S} : U \to V$ and $\mathcal{T} : V \to W$ be linear transformations. Show that
$$\operatorname{rank}(\mathcal{T} \circ \mathcal{S}) \leq \operatorname{rank}(\mathcal{T})$$
and
$$\operatorname{rank}(\mathcal{T} \circ \mathcal{S}) \leq \operatorname{rank}(\mathcal{S}).$$

(c) Let $\mathcal{S}_1 : U \to V$, $\mathcal{S}_2 : U \to V$, and $\mathcal{T} : V \to W$ be linear transformations. Show that
$$\mathcal{T} \circ (\alpha \mathcal{S}_1 + \beta \mathcal{S}_2) = \alpha \mathcal{T} \circ \mathcal{S}_1 + \beta \mathcal{T} \circ \mathcal{S}_2.$$

(d) Let $\mathcal{S} : U \to V$, $\mathcal{T}_1 : V \to W$, and $\mathcal{T}_2 : V \to W$ be linear transformations. Show that
$$(\alpha \mathcal{T}_1 + \beta \mathcal{T}_2) \circ \mathcal{S} = \alpha \mathcal{T}_1 \circ \mathcal{S} + \beta \mathcal{T}_2 \circ \mathcal{S}.$$

(e) Let $\mathcal{S}_1 : \mathbb{R}^3 \to \mathbb{R}^2$, $\mathcal{S}_2 : \mathbb{R}^3 \to \mathbb{R}^2$, $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
$$\mathcal{S}_1(x, y, z) = (y, x + z), \quad \mathcal{S}_2(x, y, z) = (2z, x - y), \quad \mathcal{T}(x, y) = (y, 2x)$$
respectively. Find the formula that defines $\mathcal{T}^2 \circ (3\mathcal{S}_1 - 5\mathcal{S}_2)$ and state its domain and codomain. Is this a linear transformation?

2. Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Let $\mathscr{B}_V = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ and $\mathscr{B}_W = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be bases of V and W respectively, where $m = \dim(V)$ and $n = \dim(W)$.

(a) Let $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ be fixed. Show that we can define a linear transformation $\mathcal{F}_{ij} : V \to W$ such that
$$\mathcal{F}_{ij}(\mathbf{v}_i) = \mathbf{w}_j,$$
and for all $k \neq i$,
$$\mathcal{F}_{ij}(\mathbf{v}_k) = \mathbf{0}.$$

(b) Show that the set $\{\mathcal{F}_{ij} \mid i = 1, \dots, m;\ j = 1, \dots, n\}$ is a basis for $\operatorname{Hom}_{\mathbb{F}}(V, W)$.

(c) What are the values of
$$\dim(\operatorname{Hom}_{\mathbb{F}}(V, W)), \quad \dim(\operatorname{End}_{\mathbb{F}}(V)), \quad \text{and} \quad \dim(\operatorname{End}_{\mathbb{F}}(W))?$$

3. Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Let $\mathcal{T} : V \to W$ be a linear transformation. Prove or disprove the following statements.

(a) If $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$ are linearly independent, then $\mathcal{T}(\mathbf{v}_1), \dots, \mathcal{T}(\mathbf{v}_n) \in W$ are also linearly independent.

(b) If $\mathcal{T}(\mathbf{v}_1), \dots, \mathcal{T}(\mathbf{v}_n) \in W$ are linearly independent, then $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$ are also linearly independent.

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(c) If $\mathbf{v}_1, \dots, \mathbf{v}_n$ span V, then $\mathcal{T}(\mathbf{v}_1), \dots, \mathcal{T}(\mathbf{v}_n)$ span W.
(d) If $\mathcal{T}(\mathbf{v}_1), \dots, \mathcal{T}(\mathbf{v}_n)$ span W, then $\mathbf{v}_1, \dots, \mathbf{v}_n$ span V.

4. Let $A \in \mathbb{F}^{n \times n}$ be a fixed matrix and $A \neq O$, the zero matrix. Define the function $\mathcal{T}_A : \mathbb{F}^{n \times n} \to \mathbb{F}^{n \times n}$ by

$$\mathcal{T}_A(X) = AX - XA \tag{4.1}$$

for all $X \in \mathbb{F}^{n \times n}$.

(a) Show that $\mathcal{T}_A$ is a linear transformation.
(b) Show that if $A^m = O$, then $\mathcal{T}_A^{2m} = \mathcal{O}$.
(c) For any $X, Y \in \mathbb{R}^{n \times n}$, show that

$$\mathcal{T}_A(XY) = X\mathcal{T}_A(Y) + \mathcal{T}_A(X)Y.$$

(d) Let $B \in \mathbb{F}^{n \times n}$ be a fixed non-zero matrix such that

$$AB = BA.$$

Let $\mathcal{T}_B$ be defined as in (4.1). Show that

$$\mathcal{T}_A \circ \mathcal{T}_B = \mathcal{T}_B \circ \mathcal{T}_A.$$

(e) Prove that $\mathcal{T}_A = \mathcal{O}$ if and only if $A = \lambda I$ for some $\lambda \in \mathbb{F}$.
(f) Is the converse of (d) true, ie. is it true that if

$$\mathcal{T}_A \circ \mathcal{T}_B = \mathcal{T}_B \circ \mathcal{T}_A,$$

then

$$AB = BA.$$