MATH 110: LINEAR ALGEBRA
SPRING 2007/08
PROBLEM SET 2

1. Let V be a vector space over $\mathbb{F}$. Let $\mathbf{w} \in V$ be a fixed non-zero vector and $\mu \in \mathbb{F}$ be a fixed non-zero scalar.
(a) Show that the function $f : \mathbb{F} \to V$ defined by $f(\lambda) = \lambda\mathbf{w}$ is injective.
(b) Show that the function $g : V \to V$ defined by $g(\mathbf{v}) = \mu\mathbf{v}$ is bijective.
(c) Show that the function $h : V \to V$ defined by $h(\mathbf{v}) = \mathbf{v} + \mathbf{w}$ is bijective.

2. Let W_1 and W_2 be subspaces of a vector space V. The *sum* of W_1 and W_2 is the subset of V defined by
$$W_1 + W_2 = \{\mathbf{w}_1 + \mathbf{w}_2 \in V \mid \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}.$$
(a) Prove that $W_1 + W_2$ is a subspace of V.
(b) Prove that $W_1 + W_2$ is the *smallest* subspace of V containing both W_1 and W_2.
(c) Prove that $W_1 \cap W_2$ is the *largest* subspace of V contained in both W_1 and W_2.

3. Let W_1 and W_2 be subspaces of a vector space V. Show that the following statements are equivalent.
(i) $W_1 \cap W_2 = \{\mathbf{0}\}$.
(ii) If $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$ are such that $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{0}$, then $\mathbf{w}_1 = \mathbf{w}_2 = \mathbf{0}$.
(iii) If $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1' + \mathbf{w}_2'$, where $\mathbf{w}_1, \mathbf{w}_1' \in W_1$ and $\mathbf{w}_2, \mathbf{w}_2' \in W_2$, then $\mathbf{w}_1 = \mathbf{w}_1'$ and $\mathbf{w}_2 = \mathbf{w}_2'$.
If any one of these equivalent conditions holds, then $W_1 + W_2$ is written $W_1 \oplus W_2$ and is called the *direct sum* of W_1 and W_2.

4. (a) State and prove the analogue of the statements in Problem **2** for the direct sum of three or more subspaces.
(b) Let W_1, W_2, W_3 be subspaces of a vector space V. Suppose
$$W_1 \cap W_2 = W_1 \cap W_3 = W_2 \cap W_3 = \{\mathbf{0}\}.$$
Must $W_1 + W_2 + W_3$ be a direct sum?

5. Prove or provide a counter example for the following.
(a) Let
$$V_1 := \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathbb{R}^{2\times 2} \;\middle|\; a, b \in \mathbb{R} \right\},$$
$$V_2 := \left\{ \begin{bmatrix} c & d \\ d & -c \end{bmatrix} \in \mathbb{R}^{2\times 2} \;\middle|\; c, d \in \mathbb{R} \right\}.$$
Is it true that
$$\mathbb{R}^{2\times 2} = V_1 \oplus V_2?$$
(b) Let
$$W_1 := \{p(x) \in \mathbb{P}_3 \mid p(-x) = p(x) \text{ for all } x \in \mathbb{R}\},$$
$$W_2 := \{p(x) \in \mathbb{P}_3 \mid p(-x) = -p(x) \text{ for all } x \in \mathbb{R}\}.$$
Is it true that
$$\mathbb{P}_3 = W_1 \oplus W_2?$$

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