MATH 110: LINEAR ALGEBRA
SPRING 2007/08
PROBLEM SET 1 SOLUTIONS

1. Prove that the following are vector spaces over $\mathbb{R}$:

(a) polynomials of degree not more than d,

$$\mathbb{P}_d = \{a_0 + a_1 x + \cdots + a_d x^d \mid a_i \in \mathbb{R} \text{ for all } i\},$$

(b) m-by-n matrices

$$\mathbb{R}^{m\times n} = \{[a_{ij}]_{i,j=1}^{m,n} \mid a_{ij} \in \mathbb{R} \text{ for all } i, j\}.$$

The addition and scalar multiplication operations for polynomials and matrices are as defined in the lectures.

Solution. Routine.

2. Let V be a vector space over $\mathbb{R}$ with addition and scalar multiplication denoted by $+$ and $\cdot$ respectively. Let $W = V \times V = \{(\mathbf{v}_1, \mathbf{v}_2) \mid \mathbf{v}_1, \mathbf{v}_2 \in V\}$. Prove that W is a vector space over $\mathbb{C}$ with addition defined by

$$(\mathbf{u}_1, \mathbf{u}_2) \boxplus (\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{v}_1, \mathbf{u}_2 + \mathbf{v}_2)$$

for all $(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \in W$ and scalar multiplication defined by

$$(a + bi) \boxdot (\mathbf{v}_1, \mathbf{v}_2) = (a \cdot \mathbf{v}_1 - b \cdot \mathbf{v}_2, b \cdot \mathbf{v}_1 + a \cdot \mathbf{v}_2)$$

for all $a + bi \in \mathbb{C}$ and $(\mathbf{v}_1, \mathbf{v}_2) \in W$. Here $i = \sqrt{-1}$ and $a, b \in \mathbb{R}$.

Solution. Routine.

3. Which of the following are subspaces of $\mathbb{R}^2$? Justify your answers.

(a) $U_a = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 0,\ x, y \in \mathbb{R}\}$,
(b) $U_b = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 0,\ x, y \in \mathbb{R}\}$,
(c) $U_c = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y = 0,\ x, y \in \mathbb{R}\}$,
(d) $U_d = \{(x, y) \in \mathbb{R}^2 \mid x - y = 0,\ x, y \in \mathbb{R}\}$,
(e) $U_e = \{(x, y) \in \mathbb{R}^2 \mid x - y = 1,\ x, y \in \mathbb{R}\}$.

If we replace $\mathbb{R}$ by $\mathbb{C}$ and $\mathbb{R}^2$ by $\mathbb{C}^2$ above, will any of your answers change?

Solution. Note that $U_a = \{(0, 0)\}$ and so is a subspace. Let $\alpha, \beta \in \mathbb{R}$. U_d is a subspace by Theorem **1.8**: if $x_1 - y_1 = 0$ and $x_2 - y_2 = 0$, then $(\alpha x_1 + \beta x_2) - (\alpha y_1 + \beta y_2) = \alpha(x_1 - y_1) + \beta(x_2 - y_2) = 0$. $(0, 0) \notin U_e$ and so it is not a subspace. Note that $(1, 1), (-1, 1) \in U_b$ (resp. U_c) but $(1, 1) + (-1, 1) = (0, 2) \notin U_b$ (resp. U_c) and so U_b (resp. U_c) is not a subspace. Over $\mathbb{C}$, U_a is not a subspace since $(i, 1), (-i, 1) \in U_a$ but $(i, 1) + (-i, 1) = (0, 2) \notin U_a$.

4. Which of the following are subspaces of $\mathbb{P}_3$? Justify your answer. Here $\mathbb{P}_3$ denotes the vector space of polynomials of degree not more than 3, ie.

$$\mathbb{P}_3 = \{p(x) = a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R}\}.$$

(a) $V_a = \{p(x) \in \mathbb{P}_3 \mid \text{degree of } p(x) \text{ is } 2\}$,
(b) $V_b = \{p(x) \in \mathbb{P}_3 \mid p(0) = p(1)\}$,
(c) $V_c = \{p(x) \in \mathbb{P}_3 \mid p(0) = 1\}$,
(d) $V_d = \{p(x) \in \mathbb{P}_3 \mid p(1) = 0\}$,
(e) $V_e = \{p(x) \in \mathbb{P}_3 \mid p(x) \geq 0 \text{ for all } x \text{ with } -1 \leq x \leq 1\}$,
(f) $V_f = \{p(x) \in \mathbb{P}_3 \mid p(-x) = -p(x) \text{ for all } x\}$.

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Solution. The zero polynomial is not in V_a and V_c and so these are not subspaces. V_e is not a subspace since the polynomial $p(x) = x$ does not have an additive inverse ($-p(x) = -x \notin V_e$). Let $\alpha, \beta \in \mathbb{R}$. V_b is a subspace by Theorem **1.8**: if $p(0) = p(1)$ and $q(0) = q(1)$, then $(\alpha p + \beta q)(0) = \alpha p(0) + \beta q(0) = \alpha p(1) + \beta q(1) = (\alpha p + \beta q)(1)$. V_d is a subspace by Theorem **1.8**: if $p(1) = 0$ and $q(1) = 0$, then $(\alpha p + \beta q)(1) = \alpha p(1) + \beta q(1) = \alpha \cdot 0 + \beta \cdot 0 = 0$. V_f is a subspace by Theorem **1.8**: if $p(-x) = -p(x)$ and $q(-x) = -q(x)$, then $(\alpha p + \beta q)(-x) = \alpha p(-x) + \beta q(-x) = -\alpha p(x) - \beta q(x) = -(\alpha p + \beta q)(x)$.

5. Which of the following are subspaces of $\mathbb{R}^{2\times 2}$? Justify your answer. Here $\mathbb{R}^{2\times 2}$ denotes the vector space of 2×2 matrices, ie.

$$\mathbb{R}^{2\times 2} = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}.$$

(a) $W_a = \{A \in \mathbb{R}^{2\times 2} \mid A^2 = A\}$,
Solution. Not a subspace. $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \in W_a$ but $2I = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right] \notin W_a$.

(b) $W_b = \{A \in \mathbb{R}^{2\times 2} \mid AB = BA\}$ where $B \in \mathbb{R}^{2\times 2}$ is a fixed matrix,
Solution. Clearly $I \in W_b$ and so $W_b \neq \varnothing$. Let $A_1, A_2 \in W_b$ and $\lambda, \mu \in \mathbb{R}$. Since $A_1 B = BA_1$ and $A_2 B = BA_2$, we have

$$\begin{aligned} (\lambda A_1 + \mu A_2)B &= \lambda A_1 B + \mu A_2 B \\ &= \lambda B A_1 + \mu B A_2 \\ &= B(\lambda A_1 + \mu A_2), \end{aligned}$$

and so $\lambda A_1 + \mu A_2 \in W_b$. Hence W_b is a subspace by Theorem **1.8**.

(c) $W_c = \{A \in \mathbb{R}^{2\times 2} \mid \det(A) = \alpha\}$ where $\alpha \in \mathbb{R}$ is a fixed scalar,
Solution. For $\alpha \neq 0$, let $A \in W_c$. Observe that $\det(2A) = 4\det(A) = 4\alpha \neq \alpha$, and so $2A \notin W_c$. So W_c is not a subspace for $\alpha \neq 0$. For $\alpha = 0$, consider $A_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Since $A_1 + A_2 = I$ and $\det(I) = 1 \neq 0$, $A_1 + A_2 \notin W_c$. So W_c is not a subspace for $\alpha = 0$ either.

(d) $W_d = \{A \in \mathbb{R}^{2\times 2} \mid \operatorname{tr}(A) = \alpha\}$ where $\alpha \in \mathbb{R}$ is a fixed scalar,
Solution. For $\alpha \neq 0$, let $A \in W_d$. Observe that $\operatorname{tr}(2A) = 2\operatorname{tr}(A) = 2\alpha \neq \alpha$, and so $2A \notin W_d$. So W_d is not a subspace for $\alpha \neq 0$. For $\alpha = 0$, let $A_1, A_2 \in W_d$ and $\lambda, \mu \in \mathbb{R}$. Since $\operatorname{tr}(A_1) = 0 = \operatorname{tr}(A_2)$, we have

$$\operatorname{tr}(\lambda A_1 + \mu A_2) = \lambda \operatorname{tr}(A_1) + \mu \operatorname{tr}(A_2) = \lambda \cdot 0 + \mu \cdot 0 = 0,$$

and so $\lambda A_1 + \mu A_2 \in W_d$. Hence W_d is a subspace by Theorem **1.8**.

(e) $W_e = \{A \in \mathbb{R}^{2\times 2} \mid A\mathbf{x} = \mathbf{b}\}$ where $\mathbf{x}, \mathbf{b} \in \mathbb{R}^2$ are two fixed vectors,
Solution. For $\mathbf{b} \neq \mathbf{0}$, let $A \in W_e$. Observe that $2A\mathbf{x} = 2\mathbf{b} \neq \mathbf{b}$, and so $2A \notin W_e$. So W_e is not a subspace for $\mathbf{b} \neq \mathbf{0}$. For $\mathbf{b} = \mathbf{0}$, let $A_1, A_2 \in W_e$ and $\lambda, \mu \in \mathbb{R}$. Since $A_1\mathbf{x} = \mathbf{0} = A_2\mathbf{x}$, we have

$$(\lambda A_1 + \mu A_2)\mathbf{x} = \lambda A_1\mathbf{x} + \mu A_2\mathbf{x} = \lambda\mathbf{0} + \mu\mathbf{0} = \mathbf{0},$$

and so $\lambda A_1 + \mu A_2 \in W_e$. Hence W_e is a subspace by Theorem **1.8**.

(f) $W_f = \{A \in \mathbb{R}^{2\times 2} \mid A\mathbf{x} = \lambda\mathbf{x} \text{ for some } \lambda \in \mathbb{R}\}$ where $\mathbf{x} \in \mathbb{R}^2$ is a fixed vector.
Solution. Let $A_1, A_2 \in W_f$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then there exists λ_1 and λ_2 such that $A_1\mathbf{x} = \lambda_1\mathbf{x}$ and $A_2\mathbf{x} = \lambda_2\mathbf{x}$. So

$$\begin{aligned} (\alpha_1 A_1 + \alpha_2 A_2)\mathbf{x} &= \alpha_1 A_1\mathbf{x} + \alpha_2 A_2\mathbf{x} \\ &= \alpha_1\lambda_1\mathbf{x} + \alpha_2\lambda_2\mathbf{x} \\ &= (\alpha_1\lambda_1 + \alpha_2\lambda_2)\mathbf{x} \end{aligned}$$

and so $\alpha_1 A_1 + \alpha_2 A_2 \in W_f$. Hence W_f is a subspace by Theorem **1.8**.

6. Let V be a vector space over a field $\mathbb{F}$ and W be a subspace of V.

(a) Let $\mathbf{0}_V$ be the additive identity of V and $\mathbf{0}_W$ be the additive identity of W. Prove that $\mathbf{0}_V = \mathbf{0}_W$.
SOLUTION. By definition, $\mathbf{0}_V + \mathbf{v} = \mathbf{v} = \mathbf{v} + \mathbf{0}_V$ for all $\mathbf{v} \in V$. Since $W \subseteq V$, we have that

$$\mathbf{0}_V + \mathbf{w} = \mathbf{w} = \mathbf{w} + \mathbf{0}_V$$

for all $\mathbf{w} \in W$. In other words, $\mathbf{0}_V$ is an additive identity of W. By the uniqueness of additive identity (Theorem **1.1**) applied to the vector space W, it follows that $\mathbf{0}_V = \mathbf{0}_W$

(b) Let $\mathbf{w} \in W$. So $\mathbf{w} \in V$ in particular. Let $\mathbf{v} \in V$ be the additive inverse of $\mathbf{w}$ as an element of V. Let $\mathbf{v}' \in W$ be the additive inverse of $\mathbf{w}$ as an element of W. Prove that $\mathbf{v} = \mathbf{v}'$.
SOLUTION. By our choice of $\mathbf{v}$ and $\mathbf{v}'$, $\mathbf{v} + \mathbf{w} = \mathbf{0}_V$ and $\mathbf{v}' + \mathbf{w} = \mathbf{0}_W$. Since $\mathbf{0}_V = \mathbf{0}_W$, we have that

$$\mathbf{v} + \mathbf{w} = \mathbf{v}' + \mathbf{w}.$$

Adding $\mathbf{v}$ to both sides of the equation and using additive associativity yields

$$\mathbf{v} + (\mathbf{w} + \mathbf{v}) = \mathbf{v}' + (\mathbf{w} + \mathbf{v}),$$

so

$$\mathbf{v} + \mathbf{0}_V = \mathbf{v}' + \mathbf{0}_V,$$

and so

$$\mathbf{v} = \mathbf{v}',$$

as required.