MATH 110: LINEAR ALGEBRA
SPRING 2007/08
PROBLEM SET 1

In all your proofs, state clearly which field axioms or vector space axioms you have used to get from one step to the next. You may wish to underline the vectors (ie. elements of the vector space) to distinguish them from the scalars (ie. elements of the field) — e.g. 0 denotes the additive identity of the field while $\underline{0}$ denotes the additive identity of the vector space.

1. Prove that the following are vector spaces over $\mathbb{R}$:

(a) polynomials of degree not more than d,

$$\mathbb{P}_d = \{a_0 + a_1 x + \cdots + a_d x^d \mid a_i \in \mathbb{R} \text{ for all } i\},$$

(b) m-by-n matrices

$$\mathbb{R}^{m\times n} = \{[a_{ij}]_{i,j=1}^{m,n} \mid a_{ij} \in \mathbb{R} \text{ for all } i, j\}.$$

The addition and scalar multiplication operations for polynomials and matrices are as defined in the lectures.

2. Let V be a vector space over $\mathbb{R}$ with addition and scalar multiplication denoted by $+$ and $\cdot$ respectively. Let $W = V \times V = \{(\mathbf{v}_1, \mathbf{v}_2) \mid \mathbf{v}_1, \mathbf{v}_2 \in V\}$. Prove that W is a vector space over $\mathbb{C}$ with addition defined by

$$(\mathbf{u}_1, \mathbf{u}_2) \boxplus (\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{v}_1, \mathbf{u}_2 + \mathbf{v}_2)$$

for all $(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \in W$ and scalar multiplication defined by

$$(a + bi) \boxdot (\mathbf{v}_1, \mathbf{v}_2) = (a \cdot \mathbf{v}_1 - b \cdot \mathbf{v}_2, b \cdot \mathbf{v}_1 + a \cdot \mathbf{v}_2)$$

for all $a + bi \in \mathbb{C}$ and $(\mathbf{v}_1, \mathbf{v}_2) \in W$. Here $i = \sqrt{-1}$ and $a, b \in \mathbb{R}$.

3. Which of the following are subspaces of $\mathbb{R}^2$? Justify your answers.

(a) $U_a = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 0,\ x, y \in \mathbb{R}\}$,
(b) $U_b = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 0,\ x, y \in \mathbb{R}\}$,
(c) $U_c = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y = 0,\ x, y \in \mathbb{R}\}$,
(d) $U_d = \{(x, y) \in \mathbb{R}^2 \mid x - y = 0,\ x, y \in \mathbb{R}\}$,
(e) $U_e = \{(x, y) \in \mathbb{R}^2 \mid x - y = 1,\ x, y \in \mathbb{R}\}$.

If we replace $\mathbb{R}$ by $\mathbb{C}$ and $\mathbb{R}^2$ by $\mathbb{C}^2$ above, will any of your answers change?

4. Which of the following are subspaces of $\mathbb{P}_3$? Justify your answer. Here $\mathbb{P}_3$ denotes the vector space of polynomials of degree not more than 3, ie.

$$\mathbb{P}_3 = \{p(x) = a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R}\}.$$

(a) $V_a = \{p(x) \in \mathbb{P}_3 \mid \text{degree of } p(x) \text{ is } 2\}$,
(b) $V_b = \{p(x) \in \mathbb{P}_3 \mid p(0) = p(1)\}$,
(c) $V_c = \{p(x) \in \mathbb{P}_3 \mid p(0) = 1\}$,
(d) $V_d = \{p(x) \in \mathbb{P}_3 \mid p(1) = 0\}$,
(e) $V_e = \{p(x) \in \mathbb{P}_3 \mid p(x) \geq 0 \text{ for all } x \text{ with } -1 \leq x \leq 1\}$,
(f) $V_f = \{p(x) \in \mathbb{P}_3 \mid p(-x) = -p(x) \text{ for all } x\}$.

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5. Which of the following are subspaces of $\mathbb{R}^{2\times 2}$? Justify your answer. Here $\mathbb{R}^{2\times 2}$ denotes the vector space of 2×2 matrices, ie.

$$\mathbb{R}^{2\times 2} = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}.$$

(a) $W_a = \{A \in \mathbb{R}^{2\times 2} \mid A^2 = A\}$,
(b) $W_b = \{A \in \mathbb{R}^{2\times 2} \mid AB = BA\}$ where $B \in \mathbb{R}^{2\times 2}$ is a fixed matrix,
(c) $W_c = \{A \in \mathbb{R}^{2\times 2} \mid \det(A) = \alpha\}$ where $\alpha \in \mathbb{R}$ is a fixed scalar,
(d) $W_d = \{A \in \mathbb{R}^{2\times 2} \mid \operatorname{tr}(A) = \alpha\}$ where $\alpha \in \mathbb{R}$ is a fixed scalar,
(e) $W_e = \{A \in \mathbb{R}^{2\times 2} \mid A\mathbf{x} = \mathbf{b}\}$ where $\mathbf{x}, \mathbf{b} \in \mathbb{R}^2$ are two fixed vectors,
(f) $W_f = \{A \in \mathbb{R}^{2\times 2} \mid A\mathbf{x} = \lambda\mathbf{x} \text{ for some } \lambda \in \mathbb{R}\}$ where $\mathbf{x} \in \mathbb{R}^2$ is a fixed vector.

6. Let V be a vector space over a field $\mathbb{F}$ and W be a subspace of V.
(a) Let $\mathbf{0}_V$ be the additive identity of V and $\mathbf{0}_W$ be the additive identity of W. Prove that $\mathbf{0}_V = \mathbf{0}_W$.
(b) Let $\mathbf{w} \in W$. So $\mathbf{w} \in V$ in particular. Let $\mathbf{v} \in V$ be the additive inverse of $\mathbf{w}$ as an element of V. Let $\mathbf{v}' \in W$ be the additive inverse of $\mathbf{w}$ as an element of W. Prove that $\mathbf{v} = \mathbf{v}'$.