MATH 110: LINEAR ALGEBRA
SPRING 2007/08
HANDOUT 1

1. Field

Let $\mathbb{F}$ be a nonempty set. Let $+$ and $\cdot$ be two binary operations on $\mathbb{F}$ called **addition** and **multiplication** respectively. Then $(\mathbb{F}, +, \cdot)$ is called a **field** if the following axioms are satisfied.

① *additive associativity*: $(a+b)+c = a+(b+c)$ for all $a, b, c \in \mathbb{F}$;
② *additive identity*: there exists $0 \in \mathbb{F}$ such that $a + 0 = a = 0 + a$ for all $a \in \mathbb{F}$;
③ *additive inverse*: for each $a \in \mathbb{F}$, there exists an element $b \in \mathbb{F}$ such that $a + b = 0 = b + a$;
④ *additive commutativity*: $a + b = b + a$ for all $a, b \in \mathbb{F}$;
⑤ *distributivity*: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in \mathbb{F}$;
⑥ *multiplicative associativity*: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathbb{F}$;
⑦ *multiplicative identity*: there exists $1 \in \mathbb{F}\backslash\{0\}$ such that $a \cdot 1 = a = 1 \cdot a$ for all $a \in \mathbb{F}$;
⑧ *multiplicative inverse*: for each $a \in \mathbb{F}\backslash\{0\}$, there exists $b \in \mathbb{F}\backslash\{0\}$ such that $a \cdot b = 1 = b \cdot a$;
⑨ *multiplicative commutativity*: $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{F}$.

These axioms are also used in defining the following algebraic structures:

AXIOMS	NAME	AXIOMS	NAME
①	*semigroup*	①–⑤	*ring*
①–②	*monoid*	①–⑥	*associative ring*
①–③	*group*	①–⑦	*associative ring with unity*
①–④	*abelian group*	①–⑧	*division ring*
		①–⑨	*field*

Remarks:

- By binary operation, we mean that closure is automatically satisfied, ie. $a+b, a\cdot b \in \mathbb{F}$ for all $a, b \in \mathbb{F}$.
- The additive inverse of $a \in \mathbb{F}$ in ③ is usually denoted $-a$.
- The multiplicative inverse of $a \in \mathbb{F}\backslash\{0\}$ in ⑧ is usually denoted a^{-1} or $1/a$.
- For simplicity, we will often denote the field by $\mathbb{F}$ instead of $(\mathbb{F}, +, \cdot)$.
- In your written work, you may write F1, F2, . . . ,F9 for ①, ②, . . . ,⑨.

2. Vector Space

Let $(\mathbb{F}, +, \cdot)$ be a field. Let V be a nonempty set. Then $(V, \oplus, \odot, \mathbb{F})$ is a vector space over $\mathbb{F}$ if the following axioms ❶–❾ are satisfied.

❶ *closure under vector addition*: $\mathbf{u} \oplus \mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$;
❷ *closure under scalar multiplication*: $a \odot \mathbf{u} \in V$ for all $a \in \mathbb{F}$ and $\mathbf{u} \in V$;
❸ *existence of zero vector*: there exists $\mathbf{0} \in V$ such that $\mathbf{u} \oplus \mathbf{0} = \mathbf{u} = \mathbf{0} \oplus \mathbf{u}$ for all $\mathbf{u} \in V$;
❹ *existence of negative vector*: for each $\mathbf{u} \in V$, there exists $\mathbf{v} \in V$ such that $\mathbf{u} \oplus \mathbf{v} = \mathbf{0} = \mathbf{v} \oplus \mathbf{u}$;
❺ *associative law for vector addition*: $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
❻ *commutative law for vector addition*: $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$;
❼ *distributive laws for scalar multiplication over vector and scalar addition*: $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})$ and $(a + b) \odot \mathbf{u} = (a \odot \mathbf{u}) \oplus (b \odot \mathbf{u})$ for all $a, b \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$;
❽ *associative law for scalar multiplication*: $(a \cdot b) \odot \mathbf{u} = a \odot (b \odot \mathbf{u})$ for all $a, b \in \mathbb{F}$ and $\mathbf{u} \in V$;
❾ *unity law for scalar multiplication*: $1 \odot \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.

Remarks:

- In this context, elements of $\mathbb{F}$ are called **scalars**.
- Elements of V are called **vectors**.
- The operations $\oplus$, $\odot$ are called **vector addition** and **scalar multiplication** respectively.

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- We will call $+$ and $\cdot$ the **field addition** and **field multiplication** operations respectively to distinguish them from vector addition and scalar multiplication.
- For simplicity, we will often denote the vector space by V instead of $(V, \oplus, \odot, \mathbb{F})$.
- Note that ❶ says that $\oplus$ is a binary operation and axioms ❺, ❸, ❹, ❻ are just axioms ①, ②, ③, ④, so $(V, \oplus)$ is an abelian group.
- In your written work, you may write V1, V2, …, V9 for ❶, ❷, …, ❾.

In symbolic form,

❶ $\mathbf{u} \oplus \mathbf{v} \in V \quad (\forall \mathbf{u}, \mathbf{v} \in V)$;
❷ $a \odot \mathbf{u} \in V \quad (\forall a \in \mathbb{F}, \forall \mathbf{u} \in V)$;
❸ $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u} \quad (\forall \mathbf{u}, \mathbf{v} \in V)$;
❹ $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) \quad (\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V)$;
❺ $\exists \mathbf{0} \in V \quad \ni \quad \mathbf{u} \oplus \mathbf{0} = \mathbf{u} = \mathbf{0} \oplus \mathbf{u} \quad (\forall \mathbf{u} \in V)$;
❻ $\forall \mathbf{u} \in V, \quad \exists \mathbf{v} \in V \quad \ni \quad \mathbf{u} \oplus \mathbf{v} = \mathbf{0} = \mathbf{v} \oplus \mathbf{u}$;
❼ $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}), \quad (a + b) \odot \mathbf{u} = (a \odot \mathbf{u}) \oplus (b \odot \mathbf{v}) \quad (\forall a, b \in \mathbb{F}, \forall \mathbf{u}, \mathbf{v} \in V)$;
❽ $(a \cdot b) \odot \mathbf{u} = a \odot (b \odot \mathbf{u}) \quad (\forall a, b \in \mathbb{F}, \forall \mathbf{u} \in V)$;
❾ $1 \odot \mathbf{u} = \mathbf{u} \quad (\forall \mathbf{u} \in V)$.