MATH 155: PROBLEM SET 6

Due November 26

1. Let $p \equiv 1 \pmod 3$. Show that you may write $p = a^2 - ab + b^2$ with $a \equiv 2 \pmod 3$, and $b \equiv 0 \pmod 3$. Give conditions $\pmod 5$ for a and b so that the equation $x^3 \equiv 5 \pmod p$ has a solution.

2. Let $\mathcal{O} = \mathbb{Z}[\omega]$ and call $\gamma \in \mathcal{O}$ primary if $\gamma \equiv 2 \pmod 3$. Show that if γ and ρ are primary then so is $-\rho\gamma$. Show that γ may be factored as $\pm\gamma_1 \cdots \gamma_s$ where the γ_j are primary primes (not necessarily distinct).

3. Factor a primary γ as in problem 2 above, and define $\chi_\gamma(\alpha) = \chi_{\gamma_1}(\alpha)\chi_{\gamma_2}(\alpha)\cdots\chi_{\gamma_s}(\alpha)$. Prove that $\chi_\gamma(\alpha) = \chi_\gamma(\beta)$ if $\alpha \equiv \beta \pmod \gamma$, and that $\chi_\gamma(\alpha\beta) = \chi_\gamma(\alpha)\chi_\gamma(\beta)$. If ρ is primary show that $\chi_\rho(\alpha)\chi_\gamma(\alpha) = \chi_{-\rho\gamma}(\alpha)$.

4. If γ and ρ are primary numbers (not necessarily prime), prove that $\chi_\gamma(\rho) = \chi_\rho(\gamma)$.

5. Prove that every odd prime number must split in at least one of the three fields $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{-2}$.

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