MATH 152: PROBLEM SET 6

Due November 13

1. In class we defined multiplicative or Dirichlet characters $\pmod q$. In this problem you will find the analogous additive characters $\pmod q$. These are functions $\psi : \mathbb{Z} \to \mathbb{C}$, not identically zero, such that ψ is periodic with period q (i.e. $\psi(n+q) = \psi(n)$) and also $\psi(m+n) = \psi(m)\psi(n)$. Describe all the additive characters $\pmod q$. Formulate and prove the orthogonality relations for these additive characters.

2. Let $a(1)$, $a(2)$, $\ldots$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} a(n)$ diverges. Suppose that the power series

$$f(z) = \sum_{n=1}^{\infty} a(n) z^n$$

converges for all complex numbers $|z| < 1$, and that

$$\lim_{r \to 1^-} f(re^{i\theta})$$

exists, and is finite, for every $0 < \theta < 2\pi$. Prove that for any progression $a \pmod q$,

$$\sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} a(n)$$

diverges. Hint: Problem 1.

3. Let p be a prime and let χ be a non-principal Dirichlet character $\pmod p$. Define the order of χ to be the least exponent ℓ such that $\chi^\ell = \chi_0$ where χ_0 is the principal character. If χ has order ℓ show that the values $\chi(n)$ for $(n, q) = 1$ are ℓ-th roots of unity. If g is a primitive root $\pmod p$ then show that $\chi(g)$ is a primitive ℓ-th root of unity. If $\chi(n)$ is a primitive ℓ-th root of unity does it necessarily follow that n is a primitive root $\pmod p$? (Recall that a primitive n-th root of unity is a number of the form $e^{2\pi i a/n}$ where $(a, n) = 1$.)

4. Let p be an odd prime. How many real (as opposed to complex) characters are there $\pmod p$? How many real characters are there $\pmod {p^\alpha}$? How many real characters are there $\pmod {2^\alpha}$? How many real characters are there $\pmod q$ for a general composite number q?