MATH 155: PROBLEM SET 5

Due November 12

1. Prove the following statement explained briefly in class: In any ring R (commutative with identity elements 0, 1), an ideal I is maximal if and only if R/I is a field. Give an example of a ring R and a non-zero ideal I with R/I infinite. Give an example of a ring R which contains a prime ideal that is not maximal.

2. Let K be a number field and let $\mathcal{O}$ be its ring of integers. Suppose that $\mathcal{O}$ has class number 2. Let $\pi \in \mathcal{O}$ be an irreducible. Show that the ideal (π) is either prime, or is the product of two (not necessarily distinct) prime ideals.

3. Let $\mathcal{O}$ be the ring of integers in a number field K. Let I and J be two ideals. We say that I and J are *relatively prime* if $I + J = \mathcal{O}$ (equivalently, $1 = a + b$ with $a \in I$ and $b \in J$).

(a) Prove that any two distinct prime ideals are relatively prime.

(b) If I and J are relatively prime, prove that $IJ = I \cap J$, and prove the Chinese Remainder Theorem: Given two residue classes $\alpha \pmod{I}$ and $\beta \pmod{J}$, there is a unique residue class $\gamma \pmod{IJ}$ with $\gamma \equiv \alpha \pmod{I}$ and $\gamma \equiv \beta \pmod{J}$.

4. Let K be an imaginary quadratic field, and let $\mathcal{O}$ be its ring of integers. Let I be an ideal in $\mathcal{O}$ and let α_1, α_2 be an integral basis for I. To this basis, associate a quadratic form $f(x, y) = N(\alpha_1 x + \alpha_2 y)$. What is the discriminant of f? Suppose β_1, β_2 is another integral basis for I, and let g be the quadratic form associated to this basis. Prove that f and g are equivalent (the determinant of the transformation is allowed here to be -1, so the equivalence need not be proper).

5. Let K be a quadratic field, and let I and J be ideals with $I \subset J$. Prove that the discriminant of J divides the discriminant of I; the discriminant of an ideal is the discriminant of an integral basis for it.