MATH 152: PROBLEM SET 5

Due November 6

1. Develop the arithmetic of the ring $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} : \quad a, b \in \mathbb{Z}\}$. Show that it is a Euclidean domain. What are the primes in this ring? Explain why unique factorization holds.

2. Use your work in question 2 to describe what primes are of the form $x^2 + 2y^2$. More generally, what numbers may be written as $x^2 + 2y^2$?

3. Evaluate $L(1, \chi_8)$, $L(1, \chi_{-8})$ and $L(1, \chi_5)$.

4. Recall that $f(x) = O(g(x))$ means that there exists a constant C such that $|f(x)| \leq Cg(x)$. Here the domain of f (and g) must be understood in context, as mentioned in class. Give an example of a function f such that as $x \to \infty$, $(\log x)^n = O(f(x))$ for any $n \geq 1$, and such that $f(x) = O(x^\alpha)$ for any $\alpha > 0$. That is f grows faster than any power of $\log x$, and slower than any power of x.

5. Define the multiplicative function $\mu(n)$ (the Möbius function by setting $\mu(1) = 1$, $\mu(p) = -1$ (for primes p) and $\mu(p^k) = 0$ for $k > 1$. When does the series $\sum_{n=1}^{\infty} \mu(n) n^{-s}$ converge? Prove that in the range $s > 1$ we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

and that

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s} = \frac{\zeta(s)}{\zeta(2s)}.$$