MATH 155: PROBLEM SET 4

Due October 29

1. Prove the following statement explained briefly in class: Let F be a subfield of the complex numbers which is an n-dimensional vector space over $\mathbb{Q}$. By our definition, F is an algebraic number field. Prove that every element in F is algebraic of degree at most n.

2. Let p be a prime, and let $f : \mathbb{Z} \to \mathbb{C}$ be a periodic function with period p; that is, $f(n+p) = f(n)$ for all $n \in \mathbb{Z}$. Define the *Fourier coefficients*

$$\hat{f}(k) = \sum_{n=0}^{p-1} f(n) e^{-2\pi i k n/p}.$$

Prove the *Fourier inversion formula*

$$f(n) = \frac{1}{p} \sum_{k=0}^{p-1} \hat{f}(k) e^{2\pi i k n/p},$$

and *Parseval's formula*

$$p \sum_{n=0}^{p-1} |f(n)|^2 = \sum_{k=0}^{p-1} |\hat{f}(k)|^2.$$

Note that our Gauss sums were simply the Fourier coefficients for the Legendre symbol.

3. Let $p \equiv 3 \pmod 4$ be a prime. Show that $\mathbb{Q}(\sqrt{p}) \subset \mathbb{Q}(e^{\frac{2\pi i}{4p}})$.

4. Let α and β be algebraic integers. If γ is a root of $x^2 + \alpha x + \beta$ show that γ is an algebraic integer. (Hint: Find an appropriate $\mathbb{Z}$-module ...)

5. Let $\pi = a + b\sqrt{d}$ be an integer in $\mathbb{Q}(\sqrt{d})$ (d square-free; note π is therefore not 3.1415...), and let p be an odd prime. Prove that

$$\pi^p \equiv \begin{cases} \pi = a + b\sqrt{d} \pmod p & \text{if } \left(\frac{d}{p}\right) = 1, \\ \overline{\pi} = a - b\sqrt{d} \pmod p & \text{if } \left(\frac{d}{p}\right) = -1. \end{cases}$$

Hint: in the case $d \equiv 1 \pmod 4$ it may be easier to work with 2π.