1. Let $d_N$ denote the least common multiple of the first $N$ natural numbers 1, 2, $\ldots$, $N$. Let $\pi(x)$ denote the number of primes $p$ with $p \leq x$.

(a) What is the power of $p$ dividing $d_N$? Prove that 
\[
\log d_N = \sum_{p \leq N} \log p \left\lfloor \frac{\log N}{\log p} \right\rfloor \leq (\log N) \pi(N).
\]

(b) Let $f(x) = \sum_i a_i x^i$ be a polynomial with integer coefficients and with degree $\leq N - 1$. Prove that 
\[
\int_0^1 f(x) dx \in \mathbb{Z}.
\]

(c) Take $f_N(x) = x^N (1 - x)^N$ and use (b) to show that $d_{2N+1} \int_0^1 f_N(x) dx \geq 1$.

(d) Show that $\int_0^1 f_N(x) dx \leq 4^{-N}$ and using (c) and (a) deduce that 
\[
\pi(2N + 1) \geq \frac{(2 \log 2)N}{\log(2N + 1)}.
\]

2. The (open) disc with center $a$ and radius $r$ is the set of points $x$ with $|x - a| < r$. The unit disc of rational numbers $\{ x \in \mathbb{Q} : |x| < 1 \}$ has as its unique center the point 0. Consider now a prime $p$, and the $p$-adic unit disc $\{ x \in \mathbb{Q} : |x|_p < 1 \}$. Show that every point in this disc is at the center of the disc.

3. If $p$ is a prime prove that $(p - 1)! \equiv (p - 1) \pmod{1 + 2 + 3 + \ldots + (p - 1)}$.

4. Let $p$ be an odd prime, and let $a$ be coprime to $p$. If $a \not\equiv 1 \pmod{p}$, prove that $p$ divides $1 + a + a^2 + \ldots + a^{p-2}$.

5. Let $f(x)$ be a polynomial with integer coefficients. If $f(a) \equiv k \pmod{m}$ show that $f(a + tm) \equiv k \pmod{m}$ for all integers $t$. Using this show that $f(x)$ cannot be prime for all integer values of $x$. 

Typeset by \LaTeX