MATH 152: PROBLEM SET 2

Due October 9

1. Let d_N denote the least common multiple of the first N natural numbers 1, 2, …, N. Let $\pi(x)$ denote the number of primes p with $p \le x$.

(a) What is the power of p dividing d_N? Prove that

$$\log d_N = \sum_{p \le N} \log p \Big[\frac{\log N}{\log p}\Big] \le (\log N)\pi(N).$$

(b) Let $f(x) = \sum_i a_i x^i$ be a polynomial with integer coefficients and with degree $\le N-1$. Prove that

$$d_N \int_0^1 f(x)dx \in \mathbb{Z}.$$

(c) Take $f_N(x) = x^N(1-x)^N$ and use (b) to show that $d_{2N+1}\int_0^1 f_N(x)dx \ge 1$.

(d) Show that $\int_0^1 f_N(x)dx \le 4^{-N}$ and using (c) and (a) deduce that

$$\pi(2N+1) \ge \frac{(2\log 2)N}{\log(2N+1)}.$$

2. The (open) disc with center a and radius r is the set of points x with $|x-a| < r$. The unit disc of rational numbers $\{x \in \mathbb{Q} : \ |x| < 1\}$ has as its unique center the point 0. Consider now a prime p, and the p-adic unit disc $\{x \in \mathbb{Q} : \ |x|_p < 1\}$. Show that every point in this disc is at the center of the disc!

3. If p is a prime prove that $(p-1)! \equiv (p-1) \pmod{1+2+3+\ldots+(p-1)}$.

4. Let p be an odd prime, and let a be coprime to p. If $a \not\equiv 1 \pmod p$, prove that p divides $1 + a + a^2 + \ldots + a^{p-2}$.

5. Let $f(x)$ be a polynomial with integer coefficients. If $f(a) \equiv k \pmod m$ show that $f(a+tm) \equiv k \pmod m$ for all integers t. Using this show that $f(x)$ cannot be prime for all integer values of x.