POLYA SEMINAR WEEK 1: INDUCTION, PARITY, PIGEONHOLE

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The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. The Ritz-Carlton has 80 rooms, but 100 guests. Fortunately, at any point of time no more than 80 guests show up. At check in, the hotel manager craftily gives each guest a number of keys so that no matter which eighty guests show up the manager can assign each guest a room to which he already has a key. (For example, the manager could do this by giving each guest all eighty keys.) What is the minimum number of keys the manager can get away with?

2. Fifteen chairs are evenly placed around a circular table on which there are name cards for fifteen guests. The guests fail to notice these cards until after they sit down, and it turns out that no one is sitting in front of his own card. Prove that the table can be rotated so that at least two of the guests are simultaneously correctly seated.

3. (Engel & closely related to A1 from last year) Each of ten line segments has length between 1 cm and 55 cms. Prove that you can select three sides of a triangle among the segments.

4. Twenty three people, each of integral weight, decide to play football by separating into two teams of eleven, plus a referee. To keep things fair, the chosen teams must have equal total weight. It turns out that no matter who is chosen to be referee, this can always be done. Show that the twenty three people must all have the same weight.

5. Let $P(x)$ be a polynomial with integer coefficients and degree at most n. Suppose that $|P(x)| < n$ for all $|x| < n^2$. Show that P is constant.

6. Prove that among any seven real numbers $y_1, \ldots, y_7$ there exist two y_i and y_j with

$$0 \le \frac{y_i - y_j}{1 + y_i y_j} \le \frac{1}{\sqrt{3}}.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

7. (Engel) The vertices of a regular 7-gon are colored black and white. Show that there are three vertices of the same color which form an isosceles triangle.

8. For any natural number $N \geq 2$ prove the inequality

$$\sqrt{2\sqrt{3\sqrt{4\sqrt{\ldots\sqrt{N}}}}} < 3.$$

9. (Engel) Find a closed formula for the sequence a_n defined as follows:

$$a_1 = 1, \qquad a_{n+1} = \frac{1}{16}(1 + 4a_n + \sqrt{1 + 24a_n}).$$

10. (Engel) Prove that if n points are not all collinear then there are at least n distinct lines joining them.

11. (Engel) Color the points in the plane red or blue. Show that either for the color red we have that given any distance, there are two points colored red that distance apart, or we have this property for the color blue.

12. (A1, 1990) Let $T_0 = 2$, $T_1 = 3$, $T_3 = 6$ and for $n \geq 3$

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392.$$

Find with proof a formula for T_n of the form $T_n = A_n + B_n$ where A_n and B_n are well known sequences.

Extra problems.

13. Consider all strings of length 300 consisting of zeros and ones. You are allowed to remove any hundred bits from each string of length 300, so that you are left with a string of length 200 (which is a subsequence of your original string). Your goal is to find the smallest set of strings of length 200, such that every string of length 300 may be reduced to one of these strings by removing 100 bits. What is the best you can do? (I don't know the answer to this one!)

14. (a) Twelve jurors are seated around a circular table. They break for lunch. Upon returning they seat themselves in a different order from when they started. Prove that there must be two jurors who have the same number of people seated between them both before lunch and after lunch.

(b) What if there are eleven jurors instead of twelve?

(c) What if there are n jurors?