POLYA SEMINAR WEEK 6: PROBABILITY AND COMBINATORICS

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The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. How many permutations σ of the group S_n are there with no fixed points?

2. If one tosses a fair coin until a head first appears, what is the probability that this occurs on an even-numbered toss?

3. (Putnam and Beyond # 912) What is the probability that a randomly chosen permutation in S_n has 1 and 2 in the same cycle?

4. (Putnam and Beyond # 931) What is the probability that three randomly selected points on a circle lie on a semicircle?

5. (Putnam and Beyond # 899) Let E be a set of n elements and F a set of p elements with $p \le n$. What is the number of surjections $f : E \to F$?

6. (Putnam and Beyond # 844) Several chords are constructed in a circle of radius 1. Prove that if every diameter intersects at most k chords then the total length of the chords is $\le k\pi$.

7. (Tao and Vu's Additive Combinatorics) Let A and B be subsets of a finite abelian group G. Show that there exist x and y in G such that

$$1 - \frac{|A \cap (B+x)|}{|G|} \le \Big(1 - \frac{|A|}{|G|}\Big)\Big(1 - \frac{|B|}{|G|}\Big) \le 1 - \frac{|A \cap (B+y)|}{|G|}.$$

Here $B + x$ denotes the set $\{b + x : \ b \in B\}$ and similarly for $B + y$.

8. (Erdős, 1956) Let A be a finite set of integers. Show that there is a subset B of A with $|B| \ge |A|/3$, and such that B is *sum-free* (that is, B does not contain a solution to the equation $x + y = z$).

9. The elements in a determinant are arbitrary integers. What is the probability that the value of the determinant is odd?

Typeset by $\mathcal{AMS}$-TEX

10. (Zeitz 4.3.21) A standard die is labeled 1, 2, 3, 4, 5, 6. When two dice are rolled one obtains sums between 2 and 12 and their probabilities are easily computed; for example 2 and 12 occur with probability 1/36, and 7 with probability 1/6. Can you construct a pair of non-standard dice (possibly different from each other) with positive integers on their faces, and such that when the dice are rolled the probability of obtaining a desired sum is the same as for a normal pair of dice (thus 2 and 12 still occur with probability 1/36, 7 with probability 1/6)?

11. (Bollobás, The art of Mathematics) We have two loaded dice (not identical) with 1, ..., 6 coming up with various probabilities when each one is rolled. Is it possible that when we roll them both the sums 2, ..., 12 all come up with equal probability 1/11?

12. (Bollobás, The art of Mathematics) Alice and Bob are about to play a tennis match in which the first person to win twelve games is declared the winner. Alice will serve first, but they are pondering two different strategies for who serves: either they will alternate serves, or the winner of the previous game will serve in the next game. Alice thinks she has a 0.71 chance of holding serve, while Bob has a 0.67 chance of holding serve. What strategy should Alice choose to maximize her chance of winning?

Extra problems.

13. (Zeitz 4.3.28; IMO proposal) Let n be a positive integer having at least two distinct prime factors. Show that there is a permutation $(a_1, \dots, a_n)$ of $(1, 2, \dots, n)$ such that

$$\sum_{k=1}^{n} k \cos\left(\frac{2\pi a_k}{n}\right) = 0.$$

14. (Zeitz 4.3.29) For each positive integer n define the polynomial

$$P_n(z) = 1^3 z + 2^3 z^2 + 3^3 z^3 + \ldots + n^3 z^n.$$

How are the zeros of $P_n(z)$ located? Inside, outside, or on the unit circle $|z| = 1$?

15. (IMO 1991) Let $S = \{1, 2, \dots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers that are pairwise coprime.