POLYA SEMINAR WEEK 1: INDUCTION, PARITY, PIGEONHOLE

K. Soundararajan

The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. a) Prove that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

b) Prove that

$$2! \cdot 4! \cdots (2n)! \geq ((n+1)!)^n.$$

2. Consider nine lattice points (that is, points whose coordinates are integers) in three dimensional space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.

3. (Erdős) Given any $n+1$ integers in $[1, 2n]$ show that there exist two numbers a and b with a dividing b (denoted $a|b$).

4. (Erdős) Given any $n+1$ integers in $[1, 2n]$ show that there are two numbers a and b with their greatest common divisor being 1.

5. In any three-dimensional polyhedron show that there are two faces having the same number of edges.

6. Can an 8×8 square with two diagonally opposite corners removed be tiled using 2×1 dominos?

7. (Larson 2.6.12 modified) Can you find a set of 5 integers in $[1, 24]$ such that the sums of all subsets are different? Can you have such a set with 7 integers in $[1, 24]$? How about 6?

8. (a) (Larson 2.6.4) Prove that there exist integers a, b, c (not all zero) with $|a|, |b|, |c| < 10^6$ such that $|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$.

(b) (Putnam 1980) Prove that if a, b, c are not all zero and less than a million in size, then $|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

9. (Putnam and Beyond) A sequence of m positive integers contains exactly n distinct terms. Prove that if $2^n \le m$ then there exists a block of consecutive terms whose product is a perfect square.

10. (Erdős and Szekeres) Let $x_1, \ldots, x_{n^2+1}$ be a sequence of n^2+1 distinct real numbers. Prove that there exists a monotone (increasing or decreasing) subsequence of length $n+1$.

11. Suppose $a_1 < a_2 < \ldots < a_n \le 2n$ are such that the least common multiple of any a_i and a_j is greater than $2n$. Show that $a_1 > 2n/3$.

12. (Peter Winkler) Each resident of Dot-town carries a red or blue dot on his (or her) forehead, but if (s)he ever figures out what color it is he kills himself. Each day the residents gather; one day a stranger comes and tells them something — *anything* — non-trivial about the number of blue dots. Prove that eventually every resident kills himself. Here *non-trivial* means that there is some number of blue dots for which the statement would not have been true.

13. Consider an $m \times n$ array of real numbers. First arrange the entries in each row so that they appear in descending order. Take this new array, and arrange the entries in each column so that these appear in descending order. Prove that the rows are not messed up, and still remain in descending order.

14. (Engel; 4.60) Prove that the function $\cos(x) + \cos(x\sqrt{2})$ is not periodic.

15. Among any eighteen people, show that there are four who know each other, or four who do not know each other. (For an easier problem, show that the conclusion holds with eighteen replaced by some larger number.)