POLYA SEMINAR WEEK 5: ALGEBRA, COMPLEX NUMBERS, POLYNOMIALS

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The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. Evaluate

$$\sum_{n=1}^{\infty} 3^{n-1} \sin^3 \left(\frac{x}{3^n}\right).$$

2. A regular n-gon is inscribed in a circle of radius 1. What is the product of the lengths of all its sides and all its diagonals?

3. (Zeitz 5.4.15) Let $p(x)$ be a polynomial with integer coefficients of degree 1999 and such that $p(x) = \pm 1$ for 1999 different integer values of x. Show that $p(x)$ cannot be factored into two the product of two polynomials with integer coefficients.

4. For natural numbers n, evaluate

$$\sum_{k=0}^{n} \binom{2n}{2k},$$

and

$$\sum_{k=0}^{n} \binom{3n}{3k}.$$

5. (Larson 4.1.6) Prove that there are infinitely many natural numbers a with the following property: The number $n^4 + a$ is not prime for every natural number n.

6. (Larson 4.3.11) Let k be a positive integer. Find all polynomials P with real coefficients such that $P(P(x)) = P(x)^k$.

7. Does there exist a polynomial whose coefficients are all 0 or ± 1 and having a zero of order 2012 at 1?

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8. Let p, q and r be complex numbers and let z_1, z_2, z_3 be the roots of the equation

$$z^3 - 3pz^2 + 3qz - r = 0.$$

Prove that z_1, z_2, z_3 form an equilateral triangle if and only if $p^2 = q$.

9. Consider complex numbers z_1, z_2, z_3, thought of as points in the complex plane. Show that these points are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

10. (Zeitz 4.3.28; IMO proposal) Let n be a positive integer having at least two distinct prime factors. Show that there is a permutation $(a_1, \dots, a_n)$ of $(1, 2, \dots, n)$ such that

$$\sum_{k=1}^{n} k \cos\Big(\frac{2\pi a_k}{n}\Big) = 0.$$

11. (Zeitz 4.3.29) For each positive integer n define the polynomial

$$P_n(z) = 1^3z + 2^3z^2 + 3^3z^3 + \ldots + n^3z^n.$$

How are the zeros of $P_n(z)$ located? Inside, outside, or on the unit circle $|z| = 1$?

12. (From Apoorva Khare) Show that

$$\sum_{i=1}^{2012} \prod_{\substack{j=1 \\ j\neq i}}^{2012} \frac{j}{j-i} = 1.$$

Extra Problems.

13. Evaluate

(i)

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

(ii)

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \ldots.$$

(iii) (Harder)

$$\frac{1}{1} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \ldots.$$

Dirichlet figured out how to generalize these to infinitely many such formulae, and these are in fact useful in showing that there are infinitely many primes in arithmetic progressions.

14. (IMO 1997) Find all pairs (a, b) of integers a, $b \geq 1$ such that

$$a^{b^2} = b^a.$$

15. (IMO 1991) Let $S = \{1, 2, \dots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers that are pairwise coprime.

16. (Akshay Venkatesh) From Pascal's triangle remove the "first two diagonals" and the "last two diagonals." (The first diagonal consists of all 1's as does the last diagonal.) Sum the reciprocals of all the numbers that remain: what does this equal?

17. For $n \geq 3$ show that there are no coprime non-constant polynomials p, q, r with $p(x)^n + q(x)^n = r(x)^n$.