POLYA SEMINAR WEEK 4: RECURRENCES AND GENERATING FUNCTIONS

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The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. (Larson 5.4.12) Let p and q be real numbers with $1/p - 1/q = 1$, and $0 < p \le 1/2$. Show that

$$p + \frac{p^2}{2} + \frac{p^3}{3} + \ldots = q - \frac{q^2}{2} + \frac{q^3}{3} - \ldots.$$

2. (Johann Bernoulli, 1697(!)) Prove that

$$\int_0^1 \frac{1}{x^x} dx = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \ldots.$$

3. (Larson 5.2.15) Prove that

$$\prod_{n=0}^{\infty} (1 + x^{2^n}) = \sum_{n=0}^{\infty} x^n.$$

Interpret this statement.

4. (1992: B2) For non-negative integers n and k define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j}.$$

5. Show that the number of partitions of n into parts that are not divisible by 3 equals the number of partitions of n in which no part appears more than twice.

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6. (1989: B3) Let f be a function on $[0, \infty)$ differentiable, and satisfying $f'(x) = -3f(x) + 6f(2x)$ for all $x > 0$. Assume that $|f(x)| \le e^{-\sqrt{x}}$ for all x (so that $f(x)$ tends rapidly to 0 as x increases). For n a non-negative integer, define

$$\mu_n = \int_0^\infty x^n f(x) dx$$

(sometimes called the n-th moment of f). a) Express μ_n in terms of μ_0. b) Prove that the sequence $\mu_n 3^n/n!$ always converges, and that the limit is zero if and only if $\mu_0 = 0$.

7. Evaluate

(i)

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots.$$

(ii)

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \dots.$$

(iii) (Harder)

$$\frac{1}{1} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \dots.$$

Dirichlet figured out how to generalize these to infinitely many such formulae, and these are in fact useful in showing that there are infinitely many primes in arithmetic progressions.

8. Find a closed formula for

$$\sum_{j=0}^{n} \frac{1}{j+1} \binom{2n}{2j}.$$

9. (Larson 5.4.20) Let $B(n)$ denote the number of ones in the binary expansion of the positive integer n. Determine whether or not

$$\exp\Big(\sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)}\Big)$$

is a rational number.

10. (Zeitz 4.3.22) Alberto places N checkers in a circle, some of which are black and the others white. Betül places new checkers in between the pairs of adjacent checkers in Alberto's ring: she places a white checker between every two of the same color, and a black checker between every pair of opposite color. She then removes Alberto's original checkers leaving a new ring of N checkers. Alberto now performs the same operation on this ring of checkers, and so on. Prove that if N is a power of 2 then all the checkers will eventually be white, no matter what the initial configuration.

Extra problems.

11. (Putnam 2003, A2) Let $a_1, \dots, a_n$, and $b_1, \dots, b_n$ be non-negative real numbers. Show that

$$(a_1 \cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n} \le ((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{1/n}.$$

12. (IMO 1997) Find all pairs (a, b) of integers a, $b \geq 1$ such that

$$a^{b^2} = b^a.$$

13. Let α be a real number such that 1^α, 2^α, ... are all natural numbers. Show that α is a non-negative integer.

14. Among the numbers 2^n $(1 \leq n \leq 10^6)$ how many begin with the leading decimal digit 1? Among the numbers 2^n which leading digit appears more frequently 7 or 8?

15. Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial of one variable. What are the possible images of P? Let now $P : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial of two variables. Is it possible for the image of P to be the $(0, \infty)$?

16. (IMO 1991) Let $S = \{1, 2, \ldots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers that are pairwise coprime.

17. (Akshay Venkatesh) From Pascal's triangle remove the "first two diagonals" and the "last two diagonals." (The first diagonal consists of all 1's as does the last diagonal.) Sum the reciprocals of all the numbers that remain: what does this equal?