POLYA SEMINAR WEEK 3: ANALYSIS AND INEQUALITIES

K. Soundararajan

The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. (Larson 7.4.6) Find all positive integers n such that $3^n + 4^n + \ldots + (n+2)^n = (n+3)^n$.

2. For all positive real numbers a, b, and c show that

$$a^a b^b c^c \ge a^b b^c c^a.$$

3. Prove that for all $n \ge 1$

$$e\Big(\frac{n}{e}\Big)^n \le n! \le e\Big(\frac{n+1}{e}\Big)^{n+1}.$$

4. Let $x_1, \ldots, x_n$ be positive real numbers, and write $(x-x_1)\cdots(x-x_n) = x^n + a_1 x^{n-1} + \ldots + a_n$. Show that for $1 \le i \le n$

$$\Big(\frac{|a_i|}{\binom{n}{i}}\Big)^{\frac{1}{i}}$$

is decreasing (precisely, non-increasing).

5. Suppose that $p_1, \ldots, p_n$ are non-negative real numbers such that $\sum_{i=1}^n p_i = 1$. Prove that

$$\sum_{i=1}^n -p_i \log p_i \le \log n.$$

6. (Larson 6.4.6) A triangle has angles α, β and γ. Prove that

$$-2 \le \sin(3\alpha) + \sin(3\beta) + \sin(3\gamma) \le \frac{3\sqrt{3}}{2}.$$

7. (Larson 7.4.23) If a, b, c are the lengths of the sides of a triangle then show that

$$\frac{3}{2} \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \le 2.$$

8. (Larson 7.4.22) Show that if a, b, c are positive numbers with $a+b+c=1$ then

$$\Big(a+\frac{1}{a}\Big)^2 + \Big(b+\frac{1}{b}\Big)^2 + \Big(c+\frac{1}{c}\Big)^2 \ge \frac{100}{3}.$$

9. (Putnam 2003, A2) Let $a_1, \ldots, a_n$, and $b_1, \ldots, b_n$ be non-negative real numbers. Show that

$$(a_1\cdots a_n)^{1/n} + (b_1\cdots b_n)^{1/n} \le ((a_1+b_1)(a_2+b_2)\cdots(a_n+b_n))^{1/n}.$$

10. (IMO 1997) Find all pairs (a,b) of integers a, $b \ge 1$ such that

$$a^{b^2} = b^a.$$

11. (From Apoorva Khare) Let $f \in \mathbb{R}[x]$ be a polynomial with real coefficients such that $f(x) \ge f'(x)$ for all x. Show that $f(x) \ge 0$ for all x.

Extra problems.

12. For any natural number n let $\mathcal{S}_n$ denote the set of natural numbers m such that $\{n/m\} \ge \frac{1}{2}$; here $\{x\} = x - [x]$ denotes the *fractional part* of x. Prove that

$$\sum_{m\in\mathcal{S}_n} \phi(m) = n^2.$$

For example, if $\mathcal{S}_6 = \{4,7,8,9,10,11,12\}$ and $\phi(4)+\phi(7)+\phi(8)+\phi(9)+\phi(10)+\phi(11)+\phi(12) = 2+6+4+6+4+10+4 = 36$.

13. (From Todd Cochrane) (a) Let p be an odd prime. Show that $H = \{x^p \pmod{p^2} : (x,p) = 1\}$ is a subgroup, of order $p-1$, of the group of reduced residue classes $\pmod{p^2}$.

(b). Show that any residue class $a \pmod{p^2}$ can be expressed as a sum of at most four elements from the subgroup H.

14. Let α be a real number such that 1^α, 2^α, $\ldots$ are all natural numbers. Show that α is a non-negative integer.

15. Among the numbers 2^n $(1 \le n \le 10^6)$ how many begin with the leading decimal digit 1? Among the numbers 2^n which leading digit appears more frequently 7 or 8?

16. Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial of one variable. What are the possible images of P? Let now $P : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial of two variables. Is it possible for the image of P to be the $(0,\infty)$?

17. (IMO 2012) Let $n \ge 3$ be an integer, and let a_2, a_3, $\ldots$, a_n be positive real numbers with $a_2a_3\cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^n.$$