POLYA SEMINAR WEEK 8: MISCELLANEOUS

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The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. (1998 B1) Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

for $x>0$.

2. (2001 A1) Consider a set S and a binary operation $*$ (that is, for each a, b in S we have $a*b\in S$). Assume that $(a*b)*a=b$ for all a, $b\in S$. Prove that $a*(b*a)=b$ for all $a,b\in S$.

3. (1994 B1) Find all positive integers n that are within 250 of exactly 15 perfect squares.

4. (1997 B1) Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n evaluate

$$F_n=\sum_{m=1}^{6n-1}\min\Big(\Big\{\frac{m}{6n}\Big\},\Big\{\frac{m}{3n}\Big\}\Big).$$

5. (1990 B3) Let S be a set of 2×2 integer matrices whose entries a_ij (1) are all squares of integers, and (2) satisfy $a_{ij}\le 200$. Show that if S has more than $50387(=15^4-15^2-15+2)$ elements then it has two elements that commute.

6. (2006 B2) Show that for every set $X=\{x_1,x_2,\dots,x_n\}$ of real numbers, there exists a non-empty subset S of X and an integer m such that

$$\Big|m+\sum_{s\in S}s\Big|\le\frac{1}{n+1}.$$

7. (2007 B3) Let $x_0 = 1$ and for $n \geq 0$ let $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5}\rfloor$. In particular $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}.

8. (2003 B2) Let n be a positive integer. Starting with 1, $\frac{1}{2}$, $\frac{1}{3}$, $\dots$, $\frac{1}{n}$ form a new sequence of $n-1$ entries $\frac{3}{4}$, $\frac{5}{12}$, $\dots$, $\frac{2n-1}{2n(n-1)}$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries, and continue until the final sequence produced consists of a single number x_n. Show that $x_n < 2/n$.

9. (2003 A6) For a set S of nonnegative integers let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) with $s_1, s_2 \in S$, $s_1 \neq s_2$ and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

Extra Problems.

10. (1988 B5) For positive integers n, let M_n be the $2n+1$ by $2n+1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Find, with proof, the rank of M_n. (According to one definition, the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with nonzero determinant.)

11. (from Hieu Hy Pham) Let S be a finite set of points in the plane such that there are no three collinear points in S. For each convex polygon P whose vertices are in S, let $f(P)$ be the number of vertices of P and $g(P)$ be the number of vertices of S that do not lie inside convex polygon P. Prove that $\sum_P x^{f(P)}(1-x)^{g(P)} = 1$ for every real number x. Note that in this problem, the empty set, single point sets, and line segments joining two points are all regarded as convex polygons.

12. (from Lyuboslav Panchev) There are 2011 boxes with 1, 2, , 2011 balls in them. Two players alternate taking balls out of the boxes. On each turn the player who has to play chooses a non-empty box. If it contains odd number of balls then he takes out exactly one ball. If it contains an even number of balls he takes out one or two balls. The player who takes out the last ball from the last non-empty box wins. Who has a wining strategy and what is it?

13. (1992 B6) Let $\mathcal{M}$ be a set of real $n \times n$ matrices such that

- $I \in \mathcal{M}$ where I is the $n \times n$ identity matrix;
- If $A \in \mathcal{M}$ and $B \in \mathcal{M}$ then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;
- If $A \in \mathcal{M}$ and $B \in \mathcal{M}$ then either $AB = BA$ or $AB = -BA$;
- If $A \in \mathcal{M}$ and $A \neq I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that $\mathcal{M}$ contains at most n^2 matrices.

14. (1988 A6) If a linear transformation A on an n-dimensional vector space has $n+1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

15. (Gyujin Oh) Suppose that $|a_{ii}| > \sum_{k \neq i} |a_{ik}|$ for $1 \leq i \leq n$. Prove that $A = (a_{ij})$ is invertible.

16. (1985 B6) Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} \operatorname{tr}(M_i) = 0$, where $\operatorname{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.

17. (2008 B6 : Not necessarily linear algebra problem, but this has nice and cool solution!) Let n and k be positive integers. Say that a permutation σ of

$\{1, 2, \ldots, n\}$ is k**-limited** if $|\sigma(i) - i| \leq k$ for all i. Prove that the number of k-limited permutations of $\{1, 2, \ldots, n\}$ is odd if and only if $n \equiv 0$ or $1 \pmod{2k+1}$.