POLYA SEMINAR WEEK 7: LINEAR ALGEBRA

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The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. (2009 A3) Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

2. (1988 B5) For positive integers n, let M_n be the $2n+1$ by $2n+1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Find, with proof, the rank of M_n. (According to one definition, the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with nonzero determinant.)

3. (Gyujin Oh) Let A and B be $m \times n$ and $n \times m$ matrices respectively. If we denote the characteristic polynomial of square matrix M as $p_M(t)$, then prove that

$$t^n p_{AB}(t) = t^m p_{BA}(t).$$

4. (2005 A4) Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix all of whose entries are 1. Show that $ab \le n$.

5. (Gyujin Oh) Suppose that $|a_{ii}| > \sum_{k\neq i} |a_{ik}|$ for $1 \le i \le n$. Prove that $A = (a_{ij})$ is invertible.

6. (Gyujin Oh) Let A, B be square matrices of same odd size. Prove that if $AB = 0$ then either $A + A^T$ or $B + B^T$ is not invertible.

7. (1985 B6) Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \le i \le r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} \operatorname{tr}(M_i) = 0$, where

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$\mathrm{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.

8. (2008 B6 : Not necessarily linear algebra problem, but this has nice and cool solution!) Let n and k be positive integers. Say that a permutation σ of $\{1, 2, \ldots, n\}$ is **k-limited** if $|\sigma(i) - i| \leq k$ for all i. Prove that the number of k-limited permutations of $\{1, 2, \ldots, n\}$ is odd if and only if $n \equiv 0$ or $1 \pmod{2k+1}$.

9. (1988 A6) If a linear transformation A on an n-dimensional vector space has $n+1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

Extra Problems.

10. (2006 B6) Let $k \geq 1$ be an integer. Let $a_0 > 0$ and define for $n \geq 0$

$$a_{n+1} = a_n + \frac{1}{a_n^{1/k}}.$$

Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

11. (related to 1974, A4) Let S_n denote the sum of n independent random variables taking the values ± 1 with equal probability. Show that the expected value of $|S_n|$ is

$$n2^{1-n}\binom{n-1}{\lfloor (n-1)/2 \rfloor}.$$

12. (from Hieu Hy Pham) Let S be a finite set of points in the plane such that there are no three collinear points in S. For each convex polygon P whose vertices are in S, let $f(P)$ be the number of vertices of P and $g(P)$ be the number of vertices of S that do not lie inside convex polygon P. Prove that $\sum_P x^{f(P)}(1-x)^{g(P)} = 1$ for every real number x. Note that in this problem, the empty set, single point sets, and line segments joining two points are all regarded as convex polygons.

13. (from Lyuboslav Panchev) There are 2011 boxes with 1, 2, , 2011 balls in them. Two players alternate taking balls out of the boxes. On each turn the player who has to play chooses a non-empty box. If it contains odd number of balls then he takes out exactly one ball. If it contains an even number of balls he takes out one or two balls. The player who takes out the last ball from the last non-empty box wins. Who has a wining strategy and what is it?

14. (1992 B6) Let $\mathcal{M}$ be a set of real $n \times n$ matrices such that

- $I \in \mathcal{M}$ where I is the $n \times n$ identity matrix;
- If $A \in \mathcal{M}$ and $B \in \mathcal{M}$ then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;
- If $A \in \mathcal{M}$ and $B \in \mathcal{M}$ then either $AB = BA$ or $AB = -BA$;
- If $A \in \mathcal{M}$ and $A \neq I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that $\mathcal{M}$ contains at most n^2 matrices.

15. (1999 B5) For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$ where I is the $n \times n$ identity matrix and $A = (a_{j,k})$ has entries $a_{j,k} = \cos(j\theta + k\theta)$ for all j, k.

16. (Moor Xu; from Putnam and Beyond and on Stanford Math Tournament 2010) Compute for $a > 1$

$$\int_0^{\pi} \ln(1 - 2a\cos x + a^2)\,dx.$$

17. (Moor Xu) Given that $f(x) = x^6 - 9x^2 - 6x$ has three critical points find the parabola passing through those three points.