# POLYA SEMINAR WEEK 6: ANALYSIS & INEQUALITIES

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**The Rules.** There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

**General problem solving strategies.** Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. Let $f : [0,1] \to (0,1)$ be continuous. Show that the equation

$$2x - \int_0^1 f(t)dt = 1$$

has exactly one solution in the interval $[0,1]$.

2. (Descartes' rule of signs) Let $f$ be a polynomial of degree $n$ with real coefficients. Prove that the number of positive roots of the polynomial is at most the number of sign changes between consecutive nonzero coefficients (the coefficients are arranged from highest degree to constant term), and has the same parity (as the number of sign changes).

3. Let $f : [0,1] \to \mathbb{R}$ be differentiable and suppose that there is no point $x \in [0,1]$ with $f(x) = f'(x) = 0$. Prove that $f$ has only finitely many zeros in $[0,1]$.

4. (2009 A2) Functions $f$, $g$, $h$ are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$f' = 2f^2gh + \frac{1}{gh}, \qquad f(0) = 1,$$

$$g' = fg^2h + \frac{4}{gh}, \qquad g(0) = 1,$$

$$h' = 3fgh^2 + \frac{1}{fg}, \qquad h(0) = 1.$$

Find an explicit formula for $f(x)$ valid in some open interval around 0.

5. (2008 A1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x,y) + f(y,z) + f(z,x) = 0$ for all real numbers $x$, $y$, and $z$. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x) - g(y)$ for all real numbers $x$ and $y$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

6. (2008 B5) Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number $q$, the number $f(q)$ is rational and has the same denominator as $q$. (The denominator of $q$ is the unique positive integer $b$ such that $q = a/b$ for some integer $a$ with $(a, b) = 1$.)

7. (2006 B5) For each continuous function $f : [0, 1] \to \mathbb{R}$ let $I(f) = \int_0^1 x^2 f(x)dx$ and $J(f) = \int_0^1 xf(x)^2 dx$. Find the maximum value of $I(f) - J(f)$ over all such functions $f$.

8. (1991 B2) Suppose that $f$ and $g$ are non-constant, differentiable real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of numbers $x$ and $y$

$$f(x + y) = f(x)f(y) - g(x)g(y),$$

and

$$g(x + y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$ prove that $f(x)^2 + g(x)^2 = 1$ for all $x$.

9. (2004 B5) Evaluate

$$\lim_{x\to 1^-} \prod_{n=0}^{\infty} \Big(\frac{1 + x^{n+1}}{1 + x^n}\Big)^{x^n}.$$

10. Let $P_n(x)$ denote a monic polynomial of degree $n$ with real coefficients. Show that $|P_n(x)| \geq 2$ for some $x \in [-2, 2]$ and that there exist polynomials $P_n$ with $|P_n(x)| \leq 2$ for all $x \in [-2, 2]$.

**Extra Problems.**

11. (2006 B6) Let $k \geq 1$ be an integer. Let $a_0 > 0$ and define for $n \geq 0$

$$a_{n+1} = a_n + \frac{1}{a_n^{1/k}}.$$

Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

12. (Bollobas) Let $X$ be a random variable with probability density function $f : \mathbb{R} \to [0, \infty)$. Show that for $p \geq 1$ we have

$$\|f\|_\infty \|X\|_p \geq \frac{1}{2}(p + 1)^{-1/p}.$$

Here $\|f\|$ is the $L^\infty$ norm – the infimum of $t$ such that $\{x : f(x) > t\}$ has zero measure, and $\|X\|_p^p = \mathbb{E}(|X|^p) = \int_{-\infty}^{\infty} |t|^p f(t)dt$.

13. (related to 1974, A4) Let $S_n$ denote the sum of $n$ independent random variables taking the values $\pm 1$ with equal probability. Show that the expected value of $|S_n|$ is

$$n2^{1-n}\binom{n-1}{\lfloor (n-1)/2 \rfloor}.$$

14. (from Stanley, via Pak Hin Lee) Suppose $a_1 < a_2 < \cdots < a_n$ and $b_1 > b_2 > \cdots > b_n$ are two sequences of integers with $\{a_1, \cdots, a_n, b_1, \cdots, b_n\} = \{1, 2, \cdots, 2n\}$. Prove that $\sum_{i=1}^n |a_i - b_i| = n^2$.

15. (from Hieu Hy Pham) Let $S$ be a finite set of points in the plane such that there are no three collinear points in $S$. For each convex polygon $P$ whose vertices are in $S$, let $f(P)$ be the number of vertices of $P$ and $g(P)$ be the number of vertices of $S$ that do not lie inside convex polygon $P$. Prove that $\sum_P x^{f(P)}(1-x)^{g(P)} = 1$ for every real number $x$. Note that in this problem, the empty set, single point sets, and line segments joining two points are all regarded as convex polygons.

16. (from Lyuboslav Panchev) There are 2011 boxes with 1, 2, , 2011 balls in them. Two players alternate taking balls out of the boxes. On each turn the player who has to play chooses a non-empty box. If it contains odd number of balls then he takes out exactly one ball. If it contains an even number of balls he takes out one or two balls. The player who takes out the last ball from the last non-empty box wins. Who has a wining strategy and what is it?

17. (2002 B6) Let $p$ be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo $p$ to the product of polynomials of the form $ax + by + cz$.

18. (1992 B6) Let $\mathcal{M}$ be a set of real $n \times n$ matrices such that

- $I \in \mathcal{M}$ where $I$ is the $n \times n$ identity matrix;
- If $A \in \mathcal{M}$ and $B \in \mathcal{M}$ then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;
- If $A \in \mathcal{M}$ and $B \in \mathcal{M}$ then either $AB = BA$ or $AB = -BA$;
- If $A \in \mathcal{M}$ and $A \neq I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that $\mathcal{M}$ contains at most $n^2$ matrices.

19. (2000 A6) Let $f(x)$ be a polynomial with integer coefficients. Define a sequence of integers $(a_n)$ such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer $m$ for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

20. (1999 B5) For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$ where $I$ is the $n \times n$ identity matrix and $A = (a_{j,k})$ has entries $a_{j,k} = \cos(j\theta + k\theta)$ for all $j, k$.