POLYA SEMINAR WEEK 5: ALGEBRA, POLYNOMIALS, COMPLEX NUMBERS

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The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. If $x \in \mathbb{R}$ is such that $x + 1/x \in \mathbb{Z}$, then prove that $x^n + 1/x^n \in \mathbb{Z}$ for all n.

2. Let r, s, t, u be the roots of the quartic equation $x^4 + ax^3 + bx^2 + cx + d = 0$. Prove that if $rs = tu$ then $a^2 d = c^2$.

3. Prove that $x^{101} + 101x^{100} + 102$ is irreducible over $\mathbb{Z}[x]$.

4. (2003 B1) Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically.

5. Suppose $(e_i)_{i=1}^{\infty}$ and $(f_j)_{j=1}^{\infty}$ are two orthonormal sets of vectors in an infinite dimensional inner product space. Suppose, for each j, that $\sum_{i=1}^{\infty} \langle e_i, f_j \rangle^2 = 1$, and also, $\sum_n \|e_n - f_n\|^2 < \infty$. Prove, for each i, that $\sum_{j=1}^{\infty} \langle e_i, f_j \rangle^2 = 1$.

6. (1999 B5) For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$ where I is the $n \times n$ identity matrix and $A = (a_{j,k})$ has entries $a_{j,k} = \cos(j\theta + k\theta)$ for all j, k.

7. (1992 B5) Let D_n denote the determinant of the matrix

$$\begin{pmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{pmatrix}.$$

Is the set $\{\frac{D_n}{n!}\}_{n \geq 2}$ bounded?

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

8. (2002 B6) Let p be a prime number. Prove that the determinant of the matrix

$$\begin{pmatrix} x & y & z \\ x^p & y^p & z^p \\ x^{p^2} & y^{p^2} & z^{p^2} \end{pmatrix}$$

is congruent modulo p to the product of polynomials of the form $ax + by + cz$.

9. (1992 B6) Let $\mathcal{M}$ be a set of real $n \times n$ matrices such that
- $I \in \mathcal{M}$ where I is the $n \times n$ identity matrix;
- If $A \in \mathcal{M}$ and $B \in \mathcal{M}$ then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;
- If $A \in \mathcal{M}$ and $B \in \mathcal{M}$ then either $AB = BA$ or $AB = -BA$;
- If $A \in \mathcal{M}$ and $A \neq I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that $\mathcal{M}$ contains at most n^2 matrices.

10. (2000 A6) Let $f(x)$ be a polynomial with integer coefficients. Define a sequence of integers (a_n) such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

11. Suppose a, b, c and A, B, C are real numbers with $a \neq 0$ and $A \neq 0$ such that

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$$

for all real numbers x. Prove that $|b^2 - 4ac| \leq |B^2 - 4AC|$.

Extra Problems.

11. (1995 B6) For a positive real number α let $S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \dots\}$. Prove that $\mathbb{N}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.

12. (2006 B6) Let $k \geq 1$ be an integer. Let $a_0 > 0$ and define for $n \geq 0$

$$a_{n+1} = a_n + \frac{1}{a_n^{1/k}}.$$

Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}.$$

13. (Bollobas) Let X be a random variable with probability density function $f : \mathbb{R} \to [0, \infty)$. Show that for $p \geq 1$ we have

$$\|f\|_\infty \|X\|_p \geq \frac{1}{2}(p+1)^{-1/p}.$$

Here $\|f\|$ is the L^∞ norm – the infimum of t such that $\{x : f(x) > t\}$ has zero measure, and $\|X\|_p^p = \mathbb{E}(|X|^p) = \int_{-\infty}^{\infty} |t|^p f(t) dt$.

14. (related to 1974, A4) Let S_n denote the sum of n independent random variables taking the values ± 1 with equal probability. Show that the expected value of $|S_n|$ is

$$n 2^{1-n} \binom{n-1}{\lfloor (n-1)/2 \rfloor}.$$

15. (from Stanley, via Pak Hin Lee) Suppose $a_1 < a_2 < \cdots < a_n$ and $b_1 > b_2 > \cdots > b_n$ are two sequences of integers with $\{a_1, \cdots, a_n, b_1, \cdots, b_n\} = \{1, 2, \cdots, 2n\}$. Prove that $\sum_{i=1}^n |a_i - b_i| = n^2$.