# POLYA SEMINAR WEEK 4: PROBABILITY AND COMBINATORICS

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**The Rules.** There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

**General problem solving strategies.** Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. (Jordan Ellenberg, in Slate 2004). Suppose the world series is played between two teams $A$ and $B$ and team $A$ has a probability $p \geq 1/2$ of beating team $B$ in any given game. Treat different games between the teams as independent.

(a). Under the current rules (best of seven, and stopping once a team wins four) what is the expected length of the World Series, and what is the probability that team $A$ wins?

(b). Consider the Alternate World Series which ends when a team is up 3-0, 4-1, 4-2, 5-3, or 5-4. What is the expected length of the AWS, and what is the probability that team $A$ wins?

2. (2004 A1) Basketball star Shanille OKeals team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80% of $N$ , but by the end of the season, $S(N)$ was more than 80% of $N$ . Was there necessarily a moment in between when $S(N)$ was exactly 80% of $N$?

3. For a list of numbers $a_1, a_2, \dots, a_h$ let $d(a_1, \dots, a_h)$ denote the number of distinct values in the list. Show that

$$\frac{1}{n^h} \sum_{a_1, a_2, \dots, a_h = 1}^{n} \Big(1 - \frac{d(a_1, \dots, a_h)}{n}\Big) = \Big(\frac{n-1}{n}\Big)^h.$$

4. Let $v_1, \dots, v_n$ be unit vectors in $\mathbb{R}^n$. Show that there exist $\epsilon_1, \dots, \epsilon_n = \pm 1$ such that

$$\Big| \sum_{j=1}^{n} \epsilon_j v_j \Big| \leq \sqrt{n}.$$

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Show also that there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ such that

$$\Big|\sum_{j=1}^{n} \epsilon_j v_j\Big| \geq \sqrt{n}.$$

5. (related to 1974, A4) Let $S_n$ denote the sum of $n$ independent random variables taking the values $\pm 1$ with equal probability. Show that the expected value of $|S_n|$ is

$$n 2^{1-n} \binom{n-1}{\lfloor (n-1)/2 \rfloor}.$$

6. (Bollobas) Let $X$ be a random variable with probability density function $f : \mathbb{R} \to [0, \infty)$. Show that for $p \geq 1$ we have

$$\|f\|_\infty \|X\|_p \geq \frac{1}{2}(p+1)^{-1/p}.$$

Here $\|f\|$ is the $L^\infty$ norm – the infimum of $t$ such that $\{x : f(x) > t\}$ has zero measure, and $\|X\|_p^p = \mathbb{E}(|X|^p) = \int_{-\infty}^{\infty} |t|^p f(t) dt$.

7. (Erdős, 1965) Let $A$ be a finite set of integers. Show that there is a subset $B$ of $A$ with $|B| \geq |A|/3$ and such that $B$ is sum free (that is, $B$ does not contain a non-trivial solution to the equation $x + y = z$).

8. Let $G = (V, E)$ be a graph with $n$ vertices and $e$ edges. Show that $G$ contains a bipartite subgraph with at least $e/2$ edges.

9. (Bollobas) In a game of $n$ gamblers, the $i$-th gambler starts the game with $a_i$ dollars. In each round, two gamblers are selected at random and make a fair bet, and the winner gets a dollar from the loser. A gambler losing all his money leaves the table, and the game continues until one gambler has all the money. What is the expected number of rounds? What is the probability that the $i$-th gambler ends up with all the money?

10. The Ramsey number $R(k, \ell)$ is the smallest integer $n$ such that in any two coloring of the edges of a complete graph $K_n$ on $n$ vertices by red and blue there is either a red $K_k$ or a blue $K_\ell$. For example, $R(2, 2) = 6$. If

$$\binom{n}{k} < 2^{\binom{k}{2}-1},$$

then show that

$$R(k, k) > n.$$

**Extra Problems.**

11. (1995 B6) For a positive real number $\alpha$ let $S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \ldots\}$. Prove that $\mathbb{N}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.

12. (1996 A5) If $p > 3$ is prime and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \ldots + \binom{p}{k}$$

is a multiple of $p^2$.

13. (2006 B6) Let $k \geq 1$ be an integer. Let $a_0 > 0$ and define for $n \geq 0$

$$a_{n+1} = a_n + \frac{1}{a_n^{1/k}}.$$

Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$