POLYA SEMINAR WEEK 3: RECURRENCES, COMBINATORICS AND GENERATING FUNCTIONS

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The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. Let f_0 and f_1 be rational numbers, not both zero. Let $f_n = f_{n-1} + f_{n-2}$ for all $n \ge 2$. Show that f_n is unbounded as $n \to \infty$. What if f_0 and f_1 are not assumed to be rational?

2. Show that the number of partitions of n into parts that are not divisible by 3 equals the number of partitions of n in which no part appears more than twice.

3. Let t_n denote the number of permutations of an n-element set that are involutions (that is, have order 2). Find a recurrence relation for t_n.

4. (1993, A2) Let x_n $(n \ge 0)$ be a sequence of non-zero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove that there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \ge 1$.

5. A *derangement* is a permutation on n-elements having no fixed points. Let d_n denote the number of permutations in S_n which are derangements, with the convention that $d_0 = 1$. Show that

$$\sum_{n=0}^{\infty} \frac{d_n}{n!} z^n = \frac{\exp(-z)}{1-z}.$$

6. (Johann Bernoulli, 1697) Prove that

$$\int_0^1 \frac{1}{x^x} dx = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

7. (1992 B3) For any pair (x, y) of real numbers, a sequence $(a_n(x, y))$ $(n \ge 0)$ is defined as follows: $a_0(x, y) = x$, and for $n \ge 0$

$$a_{n+1}(x, y) = \frac{(a_n(x, y))^2 + y^2}{2}.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Find the area of the region of points (x, y) for which $a_n(x, y)$ converges.

8. (IMO 1987) Let $p_n(k)$ denote the number of permutations of an n-element set with exactly k fixed points. Show that

$$\sum_{k=0}^{n} kp_n(k) = n!$$

and

$$\sum_{k=0}^{n} (k-1)^2 p_n(k) = n!.$$

9. Given a natural number n, describe the class of polynomials P of degree less than n such that

$$\sum_{i=0}^{n} P(i)(-1)^i \binom{n}{i} = 0.$$

10. (2006 B6) Let $k \geq 1$ be an integer. Let $a_0 > 0$ and define for $n \geq 0$

$$a_{n+1} = a_n + \frac{1}{a_n^{1/k}}.$$

Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

Extra Problems.

11. Prove Beatty's theorem: Let α and β be positive irrational numbers satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then the sequences $\lfloor \alpha n \rfloor$, $\lfloor \beta n \rfloor$, $n \geq 1$, partition the positive integers into two disjoint sets.

12. Let $1 < a_1 \leq a_2 \leq \ldots \leq a_n$ be integers with

$$\frac{1}{a_1} + \ldots + \frac{1}{a_n} = 1.$$

Show that $a_n < 2^{n!}$. Harder problem – a_n is bounded by the n-th term of Sylvester's sequence 2, 3, 7, 43, ... where each term is obtained by multiplying all previous terms and adding 1.

13. (1995 B6) For a positive real number α let $S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \ldots\}$. Prove that $\mathbb{N}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.

14. (1996 A5) If $p > 3$ is prime and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \ldots + \binom{p}{k}$$

is a multiple of p^2.