POLYA SEMINAR WEEK 2: NUMBER THEORY

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The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. Of the numbers below 2011 which has the largest number of divisors? (Such numbers were called "highly composite" by Ramanujan who made a detailed study of their structure.)

2. Prove that for every positive integer n coprime to 10 there is a multiple of n that does not contain the digit 1 in its decimal expansion.

3. Prove that the product of any four consecutive natural numbers cannot be a perfect square. (Note: In fact, Erdős and Selfridge proved the beautiful result that no product of any number of consecutive natural numbers can be a perfect power.)

4. If $4^n + 2^n + 1$ is prime then n must be a power of 3.

5. For which positive integers n does $\sum_{j=1}^n j$ divide $\prod_{j=1}^n j$.

6. Let n, a and b be positive integers. Prove that

$$\gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1.$$

7. (1991 B1) For each $n \geq 0$ let $S(n) = n - m^2$ where m is the greatest integer with $m^2 \leq n$. Define a sequence a_n by setting $a_0 = A$, $a_{k+1} = a_k + S(a_k)$. For which choices of A is this sequence eventually constant?

8. (2010 A4) Prove that for each positive number n

$$10^{10^{10^n}} + 10^{10^n} + 10^n - 1$$

is not prime.

9. Prove Beatty's theorem: Let α and β be positive irrational numbers satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then the sequences $\lfloor \alpha n \rfloor$, $\lfloor \beta n \rfloor$, $n \geq 1$, partition the positive integers into two disjoint sets.

Typeset by $\mathcal{AMS}$-TEX

10. Show that 2002^{2002} may be written as the sum of four perfect cubes, but may not be written as the sum of three perfect cubes.

11. Prove that if n is a positive integer that is divisible by at least two primes, then there exists an n-gon with all angles equal and with side lengths the numbers $1, 2, ..., n$ in some order.

Extra Problems.

12. (Related to 3 above) Prove that the product of eight consecutive natural numbers is never a perfect fourth power.

13. Let $1 < a_1 \leq a_2 \leq \ldots \leq a_n$ be integers with

$$\frac{1}{a_1} + \ldots + \frac{1}{a_n} = 1.$$

Show that $a_n < 2^{n!}$. Harder problem – a_n is bounded by the n-th term of Sylvester's sequence 2, 3, 7, 43, ... where each term is obtained by multiplying all previous terms and adding 1.

14. (1995 B6) For a positive real number α let $S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \ldots\}$. Prove that $\mathbb{N}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.

15. (1996 A5) If $p > 3$ is prime and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \ldots + \binom{p}{k}$$

is a multiple of p^2.