POLYA SEMINAR WEEK 1: INDUCTION, PARITY, PIGEONHOLE

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The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. (Car Talk Puzzler) RAY: At certain jungle prison, there are 30 prisoners who have been sentenced to be executed. Yikes! The warden, of course, has the power to pardon, and he decides to play a little game. Of course he does, right?

He announces to the prisoners that he will stand all of them in a straight line, with Prisoner Number 1 facing the wall, and the remaining prisoners lined up behind him, with each one able to see the heads of those prisoners in front of him, but not his own head of course, or those heads behind him. Kinda like standing in line at the men's room at the ol' ballpark.

Next he will place either a white or a black hat on each prisoner's head, starting at the back of the line, and a guard will then ask each prisoner, starting at the back of the line, we'll call this Prisoner Number 30, to identify the color of his own hat.

Remember, Prisoner Number 30, the one at the back of the line, can see all the other hats and prisoners in front of him. Number 29 can see the 28 hats in front of him, et cetera, et cetera, et cetera, but Prisoner Number 1 can see what?

TOM: Bupkis!

RAY: Exactly right. Of course, each prisoner can hear each of his fellow prisoners attempt to identify the correct color of his hat. When asked the color of his hat, a prisoner can say only one of two words: black, or white. That's it. Now he explains this to the prisoners, and immediately they come back and say, "Hey, come on man, give us a fighting chance! If we've only got a 50/50 chance of making it here, give us something we can do to help improve the odds."

He says, "Alright, alright, look, I hate to execute complainers, so I'll give all of you one hour to talk among yourselves and come up with a strategy. After that, silence. And remember, no funny business. Each one of you is allowed, when asked, to utter one word and one word only, either black or white. If I suspect anything is amiss, you all die."

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So the question is, is there a strategy that they can use that will improve their chances, and if so, what is it?

Notice, by the way, that I did not say how many black or how many white hats there were; it doesn't matter.

2. For all $n \in \mathbb{N}$ show that

$$\frac{1}{1} - \frac{1}{2} + \ldots - \frac{1}{2n} = \frac{1}{n+1} + \ldots + \frac{1}{2n}.$$

3. Prove that for a natural number $n > 2$

$$(n!)! > n[(n-1)!]^{n!}.$$

4. Prove that the equation

$$x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2 = 24$$

has no solutions in integers x, y, z.

5. (Zeckendorff's theorem) Every natural number can be expressed uniquely as a sum of Fibonacci numbers where the Fibonacci numbers used in the sum are all distinct, and no two consecutive Fibonacci numbers appear.

6. Set

$$S(n) = \sum_{k=0}^{n} \frac{1}{\binom{n}{k}}.$$

Prove that

$$S(n) = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}.$$

7. The numbers from 1 to 81 are written on the squares of a 9×9 board. Prove that there are two neighboring squares (that is, two squares that share an edge) which differ by at least 6.

8. $2n$ points are given in space and $n^2 + 1$ line segments are drawn between these points. Show that there is at least one triangle (that is, three points joined pairwise by line segments).

9. Let $P(x)$ be a polynomial with integer coefficients and degree at most n. Suppose that $|P(x)| < n$ for all $|x| < n^2$. Show that P is constant.

10. Prove that some power of 2 has a decimal expansion that begins with 2011. That is there exists n with $2^n = 2011\ldots$.

11. Prove that among any seven real numbers $y_1, \ldots, y_7$ there exist two y_i and y_j with

$$0 \le \frac{y_i - y_j}{1 + y_i y_j} \le \frac{1}{\sqrt{3}}.$$

12. Let $x_1, x_2, \ldots, x_k$ be real numbers such that the set

$$A = \{\cos(n\pi x_1) + \cos(n\pi x_2) + \ldots + \cos(n\pi x_k) : n \ge 1\}$$

is finite. Prove that all the x_i are rational numbers.

Extra Problems.

13. Let $1 < a_1 \leq a_2 \leq \ldots \leq a_n$ be integers with

$$\frac{1}{a_1} + \ldots + \frac{1}{a_n} = 1.$$

Show that $a_n < 2^{n!}$. Harder problem – a_n is bounded by the n-th term of Sylvester's sequence 2, 3, 7, 43, ... where each term is obtained by multiplying all previous terms and adding 1.

14. A closed disk of radius 1 contains seven points with mutual distance ≥ 1. Prove that the center of the disk is one of the seven points.