PROBLEM SET 4: ADDITIVE COMBINATORICS: WINTER 2007

1. Let A be a subset of the squares with $|A| = n$. Show that there is a function $f(n) \to \infty$ as $n \to \infty$ such that $|A + A| \geq nf(n)$. It is believed that $|A + A| \gg |A|^{1+c}$ for some positive constant c, but this remains a challenging open problem. (Hint: you will need a result of Fermat on squares.)

2. For every $C > 0$ show that there exists $N(C)$ such that if A is a set of integers with cardinality $\geq N(C)$, and $|A + A| \leq C|A|$ then A contains a three term progression.

3. Let $f : \mathbb{Z}/N \to \{|z| \leq 1\}$ be such that $\|f\|_{U^2} = 1$. Find all such f. Find all f with $\|f\|_{U^3} = 1$.

4. Prove that $\|f\|_{U^2} \leq \|f\|_{U^3} \leq \|f\|_{U^4} \leq \dots$.

5. Let A be a subset of $\mathbb{Z}/N$ with $|A| = \delta N$. Show that $3A$ contains a $\pmod N$ arithmetic progression of length N^c for some $c > 0$ depending only on δ.

6. Construct a set A such that $|A + A| \leq C|A|$ but $|A - A|$ is much larger than $C|A|$ (that is not bounded by some fixed constant times $C|A|$). How large can you make $A - A$?

7. Let $B \subset [1, n]$ with $|B| \geq 0.99n$. Let $\phi : B \to \mathbb{Z}$ be a 2-homomorphism. Prove that $\phi(k) = \alpha k + \beta$ for constants α and β. How far can you relax the 0.99, and still get the same conclusion?

8. Let N be a prime, and let A be a subset of $\mathbb{Z}/N$ with $|A| = \delta N$. Give a non-trivial bound for $|\hat{A}(r)|$ when $r \neq 0$. Using this, or otherwise, show that there exists a constant k depending only on δ such that $kA = \mathbb{Z}/N$. What happens if N is composite?