PROBLEM SET 3: ADDITIVE COMBINATORICS: WINTER 2007

1. From the proof of Roth's theorem it is clear that Behrend's set without 3 APs must have large Fourier coefficients. Exhibit some of these. Can you find all of the large coefficients?

2. In the decimal expansion of 2^n which appears more frequently as the leading digit: 7 or 8? Explain; the phenomenon you see is an example of Benford's law.

3. Let $\phi = (1 + \sqrt{5})/2$ denote the golden ratio. Prove that

$$\sum_{n \le N} e(n^2 \phi) \ll \sqrt{N \log N}.$$

4. Indicate what changes should be made to Vinogradov's proof in order to get three term progressions of primes. State a precise asymptotic formula for the number of three term progressions in the primes up to N.

5. Use the circle method to find an asymptotic formula for the number of ways of writing N as a sum of five squares. You may find it useful to know that the number of ways of writing n as a sum of two squares is $\ll n^\epsilon$; this will help you bound a certain fourth moment.

6. (Easier Waring's problem) Prove that for every k there exists a constant $C(k)$ such that every integer can be written as $\pm x_1^k \pm x_2^k \pm \ldots \pm x_m^k$ with $m \le C(k)$. (Hint: think about Weyl's *method*.)