PROBLEM SET 2: ADDITIVE COMBINATORICS: WINTER 2007

1. Give, either combinatorially or by working through Roth's proof, an explicit value of N such that if the numbers from $[1, N]$ are colored blue and red, there exists a monochromatic three term progression.

2. Suppose $a_1, \ldots, a_k$ are given integers with $\sum_{j=1}^k a_j = 0$. Explain why, for $N \geq N(\delta)$ large, a set of density δ in $[1, N]$ must contain a non-trivial solution to $\sum_{j=1}^k a_j x_j = 0$. (You don't have to give all the details; just enough to convince yourself.)

3. Let A, B, C, D be four subsets of $\mathbb{Z}$ of size N. Suppose there are αN^3 solutions to $a + b = c + d$ with $a \in A$, $b \in B$, $c \in C$, and $d \in D$. Show that there are at least $\alpha^4 N^3$ solutions to $a + b = c + d$ with a, b, c, and d all in A. Give examples to show why α^4 cannot be replaced by $c\alpha^{3.99}$ for any constant $c > 0$.

4. Suppose A has size N and contains at least αN^2 three term progressions. Show that it contains at least $\alpha^2 N^3$ solutions to $a+b = c+d$. Give examples to show why α^2 cannot be replaced by $c\alpha^{1.99}$ for any constant $c > 0$.

5. Suppose that $|\alpha - a/q| \leq 1/q^2$. Obtain a bound for

$$\sum_{n \leq N} e(\alpha n^3).$$

Deduce an upper bound for $\min_{m \leq M} \|\alpha m^3\|$.

6. Let A be a subset of $\mathbb{Z}/N$ of size at most $(\log N)/10$. Show that there exists $r \neq 0$ with $|\hat{A}(r)| \geq |A|/2$.