PROBLEM SET 1: ADDITIVE COMBINATORICS: WINTER 2007

1. Let A be a subset of the natural numbers such that every interval of length ℓ contains at least one element from A; in particular, the density of A is at least $1/\ell$. Show (using van der Waerden's theorem) that A contains arbitrarily long arithmetic progressions.

2. Let k, ℓ and r be given. Show that there exists $N = N(k, \ell, r)$ such that if the numbers from 1 to N are colored using r colors then there exist a and d such that $a + j$, $a + j + d$, $a + j + 2d$, $\ldots$, $a + j + (k-1)d$ is a monochromatic k term arithmetic progression for each $j = 0, 1, \ldots, \ell - 1$.

3. Let $\delta > 0$ be given. Show that there exists $N = N(\delta)$ such that any set $A \in [1, N]$ with $|A| \geq \delta N$ contains a parallelogram a, $a + b$, $a + c$, $a + b + c$. Obtain a good bound for $N(\delta)$. Can you formulate a generalization for higher dimensional parallelopipeds?

4. Generalize the argument of Schur's theorem presented in class to show that in any r coloring of the natural numbers, we may find a monochromatic set $x_1, \ldots, x_k$, $x_1 + \ldots + x_k$.

5. This exercise will show that given any real number θ and any $Q \geq 1$, there exists $n \in \mathbb{N}$ with $\|n^2\theta\| \leq 1/Q$. Divide the interval $[0, 1]$ into $2Q$ intervals of equal length, and color the natural number n depending on the interval into which the fractional part of $n^2\theta/2$ falls. Now play with monochromatic three term progressions, and deduce the stated result. Generalize the argument to find small values of $\|n^k\theta\|$.

6. Let X be a probability space and T a measure preserving transformation. A set R is called a set of recurrence if every set V of positive measure contains a point x with $T^r x \in V$ for some $r \in R$.

(a) Show that any infinite difference set (that is a set of the form $\{s_i - s_j : s_i, s_j \in S\}$ for some infinite set S of natural numbers) is a set of recurrence.

(b) Let $\mathcal{N} = \{n_1 < n_2 < \ldots\}$ be a *lacunary* set with $n_{i+1}/n_i > 2$ for each i. Show, by exhibiting an example of X and T, that $\mathcal{N}$ is not a set of recurrence. Hint: Diophantine approximation.