POLYA SEMINAR WEEK 7: MISCELLANEOUS PUTNAM PROBLEMS

Bob Hough and K. Soundararajan

The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. (1987 A1) Curves A, B, C, and D are defined in the plane as follows:

$$A = \{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\}$$

$$B = \{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\}$$

$$C = \{(x,y) : x^3 - 3xy^2 + 3y = 1\}$$

$$D = \{(x,y) : 3x^2y - 3x - y^3 = 0\}.$$

Prove that $A \cap B = C \cap D$.

2. (1991 A3) Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exit real numbers $r_1 < r_2 < \ldots < r_n$ such that $p(r_i) = 0$ for $1 \leq i \leq n$, and $p'((r_i + r_{i+1})/2) = 0$ for $1 \leq i \leq n-1$.

3. (1994 B2) For which real numbers c is there a straight line that intersects the curve $y = x^4 + 9x^3 + cx^2 + 9x + 4$ in four distinct points.

4. (1995 A1) Let S be a set of real numbers closed under multiplication (that is, if a, $b \in S$ then $ab \in S$). Let T and U be disjoint subsets of S whose union is S. Given that the product of any three (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

5. (1986 A2) What is the units digit of

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor ?$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

6. (1987 A6) For each positive integer n let $a(n)$ denote the number of zeros in the base 3 expansion of n. For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?

7. (1987 A1) Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}dx.$$

8. (1990 B3) Let S be a set of 2×2 integer matrices whose entries a_{ij} are all squares of integers, and are all below 200. Show that if $|S| \geq 50387 (= 15^4 - 15^2 - 15 + 2)$ then S has two elements that commute.

9. (1991 B5) Let p be an odd prime and let $\mathbb{Z}_p$ denote the field of integers modulo p. How many elements are in the set

$$\{x^2 : \quad x \in \mathbb{Z}_p\} \cap \{y^2 + 1 : \quad y \in \mathbb{Z}_p\}?$$

10. (1989 B2) Let S be a non-empty set with an associative operation that is left and right cancellative (that is $xy = xz$ implies $y = z$ and $yx = zx$ implies $y - z$). Suppose that for every $a \in S$ the set $\{a^n : n \in \mathbb{N}\}$ is finite. Must S be a group?

11. (1998 B6) Prove that, for any integers a, b, c, there exits a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

12. (1997 B5) Set $a_n = 2^{2^{2^{\cdots^{2}}}}$ where there are n-twos in all. Prove that for $n \geq 2$ we have

$$a_n \equiv a_{n-1} \pmod{n}.$$

13. (1997 A4) Let G be a group with identity e and let $\phi : G \to G$ be a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1 g_2 g_3 = h_1 h_2 h_3 = e$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (that is $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$).

14. (1995 B3) To each positive integer with n^2 decimal digits we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, we associate to the integer 8617 the determinant of $\begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix}$ which is 50. Find as a function of n the sum of all the determinants associated with n^2-digit numbers. (Leading digits are assumed to be non-zero; for example, for $n = 2$, there are 9000 determinants.)