POLYA SEMINAR WEEK 2: NUMBER THEORY

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The Rules. There are too many problems to consider. Pick a few problems that you find fun, and play around with them. The only rule is that you may not pick a problem that you already know how to solve: where's the fun in that?

General problem solving strategies. Try small cases; plug in smaller numbers. Search for a pattern. Draw pictures. Choose effective notation. Work in groups. Divide into cases. Look for symmetry. Work backwards. Argue by contradiction. Parity? Pigeonhole? Induction? Generalize the problem, sometimes that makes it easier. Be flexible: consider many possible approaches before committing to one. Be stubborn: don't give up if your approach doesn't work in five minutes. Ask. Eat pizza, have fun!

1. (From Sahana Vasudevan) Determine all numbers n such that $\phi(n) = n/3$.

2. Prove that 2^{n-1} divides $n!$ if and only if n is a power of 2.

3. Find the last digit of $2^{3^{4^5}}$ when written out in the usual decimal notation.

4. (1989: A1) How many primes are there which when written in base 10 (as usual) are such that their digits are alternating 1's and 0's beginning and ending with 1?

5. (Putnam and Beyond # 746) Show that for each positive integer n,

$$n! = \prod_{j=1}^{n} \mathrm{lcm}(1, 2, \dots, [n/j]).$$

6. (Larson 3.2.20) Show that if a number n divides a single Fibonacci number then it divides infinitely many Fibonacci numbers.

7. What is the greatest common divisor of the set of numbers $\{n^{13} - n : \ n \in \mathbb{Z}\}$?

8. (Larson 3.2.17) Suppose that S is a set of primes such that if a, b are in S (not necessarily distinct) then $ab + 4$ is also in S. Show that S is the empty set.

9. For any natural number n let $\mathcal{S}_n$ denote the set of natural numbers m such that $\{n/m\} \geq \frac{1}{2}$; here $\{x\} = x - [x]$ denotes the *fractional part* of x. Prove that

$$\sum_{m \in \mathcal{S}_n} \phi(m) = n^2.$$

For example, if $\mathcal{S}_6 = \{4, 7, 8, 9, 10, 11, 12\}$ and $\phi(4) + \phi(7) + \phi(8) + \phi(9) + \phi(10) + \phi(11) + \phi(12) = 2 + 6 + 4 + 6 + 4 + 10 + 4 = 36$.

10. (1991: B4) Let p be an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j} \binom{p+j}{j} \equiv 0 \pmod{p^2}.$$

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