MATH 152: MIDTERM EXAMINATION

Due October 30 in Class

The Rules

You are free to consult your class notes, and the books by Niven-Zuckerman-Montgomery, Hardy-Wright, Stark, and LeVeque placed on reserve at the library. You may use any result that we have covered so far in class, but you must state clearly what you are using. You may not discuss the exam with others, nor use the internet or any other source, with the exception of the books listed above. You should write out and sign the honor code, indicating also your acceptance of these rules. All the best!

The Problems

1. (15 points) (i) If $x \in \mathbb{Q}$ show that $x \in \mathbb{Z}$ if and only if $|x|_p \le 1$ for all primes p.
(ii). If x is a non-zero rational number prove that

$$|x| \cdot \prod_p |x|_p = 1,$$

where the product is over all primes p.
(iii). Let p and q be two different prime numbers. For every $\epsilon > 0$ show that there exists a rational number x such that $|x|_p < \epsilon$, and $|x-1|_q < \epsilon$.

2. (20 points) (i) For a positive real number x let $[x]$ denote the greatest integer not exceeding x. Prove that for all positive real numbers x

$$[30x] + [x] - [15x] - [10x] - [6x] = 0, \text{ or } 1.$$

(ii). For all natural numbers n show that

$$\frac{(30n)!n!}{(15n)!(10n)!(6n)!}$$

is a natural number.

3. (15 points) (i) Let p be an odd prime number, and let k be a natural number with $(k, p-1) = \ell$. Show that there are exactly $(p-1)/\ell$ distinct reduced residue classes among 1^k, 2^k, $\dots$, $(p-1)^k \pmod p$.

(ii) Let $p \equiv 2 \pmod 3$ be a prime. Evaluate

$$\sum_{n=0}^{p-1} \left(\frac{n^3+1}{p} \right).$$

4. (15 points) Let $p > 3$ be a prime with $p \equiv 3 \pmod 4$, and suppose that $q = 2p+1$ is also prime. Prove that $2^p - 1$ is composite.

5. (15 points) (i) Let n be an odd integer with $n \equiv 2 \pmod 5$. Show that the congruence $x^2 \equiv 5 \pmod n$ has no solutions.

(ii) Let n be an odd square-free integer with $n \equiv 1 \pmod 5$. Is it necessarily true that $x^2 \equiv 5 \pmod n$ has a solution? (An integer is square-free if it is not divisible by the square of any prime.)

6. (20 points) Prove that the sequence n^n is periodic $\pmod p$, where p is prime. Determine the least period. (That is, find the least positive number ℓ such that $(n+\ell)^{n+\ell} \equiv n^n \pmod p$ for all n.)