MATH 215A: MIDTERM EXAMINATION SOLUTIONS

November 1, 2011

1. Let f and g be holomorphic functions in a region (that is, connected and open) Ω, and suppose that $|f(z)| + |g(z)|$ is constant for all $z \in \Omega$. Prove that f and g are both constants.

Solution. Suppose $|f(z)| + |g(z)| = A$ for all $z \in \Omega$. Let w be a point in Ω and let D be the disk of radius r centered at w and contained entirely in Ω. By the Cauchy formula we have for any $r_0 \leq r$

$$f(w) = \frac{1}{2\pi i} \int_{|z-w|=r_0} \frac{f(z)}{z-w} dz = \frac{1}{2\pi} \int_0^{2\pi} f(w + r_0 e^{i\theta}) d\theta. \tag{1}$$

If the strict triangle inequality

$$|f(w)| < \frac{1}{2\pi} \int_0^{2\pi} |f(w + r_0 e^{i\theta})| d\theta$$

holds, then by adding to it the corresponding triangle inequality for g

$$|g(w)| \leq \frac{1}{2\pi} \int_0^{2\pi} |g(w + r_0 e^{i\theta})| d\theta,$$

we would obtain

$$A = |f(w)| + |g(w)| < \frac{1}{2\pi} \int_0^{2\pi} (|f(w + re^{i\theta})| + |g(w + re^{i\theta})|) d\theta < A,$$

which is a contradiction. Therefore we may assume that the equality

$$|f(w)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(w + r_0 e^{i\theta}) d\theta \right| = \frac{1}{2\pi} \int_0^{2\pi} |f(w + r_0 e^{i\theta})| d\theta$$

holds for all $0 < r_0 \leq r$. But this is only possible if $f(w + r_0 e^{i\theta})$ always points in the same direction; that is $f(w + r_0 e^{i\theta}) \in \mathbb{R}_{\geq 0} e^{i\phi}$ for some ϕ and all $\theta \in [0, 2\pi)$ and all $0 \leq r_0 \leq r$. The image of the open disk $|z - w| < r$ is then contained in a line, contradicting the open mapping theorem unless f is constant.

Solution 2 (suggested by work of Li Liu). In the problem we may freely multiply f and g by unimodular constants $e^{i\theta_1}$ and $e^{i\theta_2}$. In this way, we may choose a point $w \in \Omega$ and assume that both $f(w)$ and $g(w)$ are real and positive. But then $f(w) + g(w) = |f(w)| + |g(w)| = A$, and $|f(z) + g(z)| \leq |f(z)| + |g(z)| = A$ for

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX all $z \in \Omega$. That is $|f+g|$ attains a maximum at the point $w \in \Omega$ which is only possible if $f+g=A$ is constant. Moreover we have for all points $z \in \Omega$ that $|f(z)+g(z)| = |f(z)| + |g(z)|$ which is only possible if the points $f(z)$, $g(z)$ and $f(z)+g(z) = A$ are all real and positive, and gives that f and g are constants.

2. Let Ω be the region obtained by deleting from the complex plane the real line segments $[0,1]$, $[2,3]$ and $[4,5]$.

(a). Given a holomorphic function f on Ω and a cycle γ in Ω, what can you say about

$$\int_\gamma f(z)dz?$$

(b). Construct a holomorphic function f on Ω such that for any cycle γ in Ω one has

$$\int_\gamma f(z)dz = n_1 e + n_2 \pi + n_3 i,$$

for some integers n_1, n_2, and n_3.

Solution. (a) Put $I_1 = [0,1]$, $I_2 = [2,3]$ and $I_3 = [4,5]$. We can pick cycles γ_1, γ_2, γ_3 such that the γ_1 winds around points in I_1 once but not around any points in I_2 or I_3; γ_2 winds around points in I_2 once but not points around I_1 or I_3 and lastly γ_3 winds around points in I_3 once but not points around I_1 or I_2. Any cycle γ is homologous in Ω to a cycle $n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$ for some integers n_1, n_2 and n_3. By Cauchy's theorem we have

$$\int_\gamma f(z)dz = n_1 \int_{\gamma_1} f dz + n_2 \int_{\gamma_2} f dz + n_3 \int_{\gamma_3} f dz.$$

(b) Let γ_1, γ_2 and γ_3 be as above, and put

$$f(z) = \frac{1}{2\pi i}\Big(\frac{e}{z-1/2} + \frac{\pi}{z-5/2} + \frac{i}{z-9/2}\Big).$$

This is holomorphic on Ω, and we have

$$\int_{\gamma_1} f dz = e, \qquad \int_{\gamma_2} f dz = \pi, \qquad \int_{\gamma_3} f dz = i.$$

3.

(a) Let f be an entire function with $f(\sqrt{n}) = 0$ for all $n \in \mathbb{N}$, and suppose that f is not identically zero. Prove that f must have order at least 2, and that there exists such a function of order 2.

(b) If f is an entire function of order ρ then prove that f' is also an entire function of order ρ, and conversely.

Solution. (a) If f has order ρ then in the disc $|z| < R$ it can have at most $CR^{\rho+\epsilon}$ zeros. But f has at least cR^2 zeros here, and therefore $\rho \geq 2$. An example of such a function of order 2 is $\sin(\pi z^2)$.

(b). By the Cauchy formula

$$f'(w) = \frac{1}{2\pi i}\int_{|z-w|=1}\frac{f(z)}{(z-w)^2}dz,$$

and so if f has order ρ (so that $|f(z)| \le C_1 \exp(|z|^{\rho+\epsilon})$) we conclude that

$$|f'(w)| \le C_1 \exp((|w|+1)^{\rho+\epsilon}),$$

and so $f'(w)$ has order at most the order of f.

Conversely, if the order of f' is ρ, then using $f(z) = \int_0^z f'(w)dw + f(0)$, (integrate along the straight line joining 0 and z, say) we deduce that f has order at most ρ.

Thus the order of f' is bounded by the order of f and conversely, and so the two orders are equal.

4. By considering the lines of integration $\{x \in \mathbb{R}\}$, and $\{x+\pi i, x \in \mathbb{R}\}$, or otherwise, prove that (i)

$$\int_{-\infty}^{\infty}\frac{du}{e^u+e^{-u}} = \frac{\pi}{2},$$

and (ii)

$$\int_{-\infty}^{\infty}\frac{u^2du}{e^u+e^{-u}} = \frac{\pi^3}{8}.$$

Solution. (i) By the residue theorem we have

$$\int_{-\infty}^{\infty}\frac{du}{e^u+e^{-u}} = \int_{-\infty+i\pi}^{\infty+i\pi}\frac{du}{e^u+e^{-u}} + 2\pi i \operatorname*{Res}_{u=i\pi/2}\frac{1}{e^u+e^{-u}}.$$

Note that $e^u+e^{-u} = ie^{(u-i\pi/2)} - ie^{-u+i\pi/2} = 2i(u-i\pi/2)+\ldots$, and so the residue above is $1/(2i)$. Moreover, writing $u = x+i\pi$,

$$\int_{-\infty+i\pi}^{\infty+i\pi}\frac{du}{e^u+e^{-u}} = \int_{-\infty}^{\infty}\frac{dx}{-e^x-e^{-x}},$$

and so we have

$$\int_{-\infty}^{\infty}\frac{du}{e^u+e^{-u}} = -\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}} + \pi,$$

from which (i) follows.

(ii) As above we find that

$$\int_{-\infty}^{\infty}\frac{u^2du}{e^u+e^{-u}}du = \int_{-\infty+i\pi}^{\infty+i\pi}\frac{u^2}{e^u+e^{-u}}du + 2\pi i \operatorname*{Res}_{u=i\pi/2}\frac{u^2}{e^u+e^{-u}}.$$

The residue above is $(i\pi/2)^2/(2i)$, and writing $u = x+i\pi$ we have

$$\begin{aligned}\int_{-\infty+i\pi}^{\infty+i\pi}\frac{u^2}{e^u+e^{-u}}du &= \int_{-\infty}^{\infty}\frac{(x+i\pi)^2}{-e^x-e^{-x}}dx\\ &= -\int_{-\infty}^{\infty}\frac{x^2}{e^x+e^{-x}}dx - 2i\pi\int_{-\infty}^{\infty}\frac{x}{e^x+e^{-x}}dx + \pi^2\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}}.\end{aligned}$$

The middle integral is zero by symmetry, and the last term is $\pi^3/2$ by part (i). Thus we conclude that

$$\int_{-\infty}^{\infty} \frac{u^2}{e^u + e^{-u}} du = -\int_{-\infty}^{\infty} \frac{x^2}{e^x + e^{-x}} dx + \frac{\pi^3}{2} - \frac{\pi^3}{4},$$

and so (ii) follows.

5. Let $n \geq 2$ and set $P_n(z) = z^n + 3z + 1$. Show that $P_n(z)$ has exactly one zero inside the unit disc, and its remaining $n-1$ zeros lie in the annulus $1 < |z| < 4^{1/(n-1)}$.

Solution. Note that on $|z| = 1$ we have $3 = |3z| > |z^n + 1|$, and so by Rouche's theorem, $P_n(z)$ and $3z$ have the same number of zeros inside the unit disc, namely one.

On $|z| = 4^{1/(n-1)}$ note that $|z^n| = 4|z| > |3z + 1|$, and so by Rouche $P_n(z)$ and z^n have the same number of zeros in $|z| < 4^{1/(n-1)}$. This completes the proof.