MATH 215A: MIDTERM EXAMINATION

November 1, 2011

This is a closed book, closed notes test. You are free to use results proved in class, but you must state clearly the result that you are using. Complete proofs are expected for all questions, and you are also expected to write in a clear style with complete, grammatical sentences. There will be partial credit, so please include approaches to the problem, and/or heuristic reasons for what the result might be or why it might be true. All the best!

Do any four of the following five problems.

1. Let f and g be holomorphic functions in a region (that is, connected and open) Ω, and suppose that $|f(z)| + |g(z)|$ is constant for all $z \in \Omega$. Prove that f and g are both constants.

2. Let Ω be the region obtained by deleting from the complex plane the real line segments $[0, 1]$, $[2, 3]$ and $[4, 5]$.

(a). Given a holomorphic function f on Ω and a cycle γ in Ω, what can you say about

$$\int_\gamma f(z)dz?$$

(b). Construct a holomorphic function f on Ω such that for any cycle γ in Ω one has

$$\int_\gamma f(z)dz = n_1 e + n_2 \pi + n_3 i,$$

for some integers n_1, n_2, and n_3.

3.

(a) Let f be an entire function with $f(\sqrt{n}) = 0$ for all $n \in \mathbb{N}$, and suppose that f is not identically zero. Prove that f must have order at least 2, and that there exists such a function of order 2.

(b) If f is an entire function of order ρ then prove that f' is also an entire function of order ρ, and conversely.

4. By considering the lines of integration $\{x \in \mathbb{R}\}$, and $\{x + \pi i, x \in \mathbb{R}\}$, or otherwise, prove that (i)

$$\int_{-\infty}^{\infty} \frac{du}{e^u + e^{-u}} = \frac{\pi}{2},$$

and (ii)

$$\int_{-\infty}^{\infty} \frac{u^2 du}{e^u + e^{-u}} = \frac{\pi^3}{8}.$$

5. Let $n \geq 2$ and set $P_n(z) = z^n + 3z + 1$. Show that $P_n(z)$ has exactly one zero inside the unit disc, and its remaining $n-1$ zeros lie in the annulus $1 < |z| < 4^{1/(n-1)}$.