Math 215 Problem Set 3

Solutions

October 31, 2011

1 2.6.10

We show that every continuous function that can be uniformly approximated by polynomials is holomorphic. Suppose p_n converges uniformly to f. Then for any triangle T we have $\int_T f(z)dz = \int_T \lim_{n\to\infty} p_n(z)dz = \lim_{n\to\infty} \int_T p_n(z)dz = 0$ by the uniform convergence. Morera's theorem shows not that f is holomorphic.

2 Problem 2.7.4

First we prove that there is at least one bounded component of K^c. K is bounded since it is compact, which means that it is contained in a disk of sufficiently large radius. This means that the complement of this disc lies in one connected component of K^c and thus there is at most one unbounded component of K^c and since it is disconnected we get that there is at least one bounded one.

Let z_0 be an arbitrary point in one of the bounded components S of K^c. Let $f = \frac{1}{z-z_0}$. Obviously f is holomorphic in a neighborhood of K. Suppose f can be approximated uniformly with polynomials. This means that there is a polynomial p such that $|f(z) - p(z)| < 1$ for all $z \in K \Rightarrow |(z-z_0)p(z) - 1| < 1$. Consider $g(z) = (z-z_0)p(z) - 1$, then g is holomorphic in the entire complex plane and $|g(z)| < 1$ for each $z \in K$. Since $\partial S \subset \partial K^c \subset \partial K \subset K$ we have that $|g(z)| < 1$ for each $z \in \partial S$, since S is a connected domain from the maximum modulus principle we have that $|g(z)| < 1$ for each $z \in S$ which is a contradiction since $z_0 \in S$ and $g(z_0) = 1$.

3 Problem 3.8.18

We want to show that $\int_C \frac{f(z) - f(\xi)}{z - \xi} d\xi = 0$. The holomorphicity of f implies that $\frac{f(z)-f(\xi)}{z-\xi}$ is bounded when ξ is close to z since it is closed to $f'(z)$. Suppose $|\frac{f(z)-f(\xi)}{z-\xi}| < A$. Now the circle $C(z,\epsilon)$ is holomorphic to C in $\mathbf{C} - z$ and therefore $|\int_C \frac{f(z)-f(\xi)}{z-\xi} d\xi| = |\int_{C(z,\epsilon} \frac{f(z)-f(\xi)}{z-\xi} d\xi| \leq \int_C |\frac{f(z)-f(\xi)}{z-\xi}| d\xi \leq 2\pi\epsilon$ and this can be made arbitrary small.

4 Problem 3.8.21

(b) Let $\gamma : [0,1] \to \Omega$ be a curve. We show it is homotopic to the constant curve $\gamma' : [0,1] \to \Omega$ given by $\gamma'(x) = z_0$. The function $H : [0,1] \times [0 : 1] \to \Omega$ defined by $H(x,t) = t\gamma(x) + (1-t)z_0$ is clearly continuous and gives a homotopy between γ and γ'. We need the star-shapedness of Ω to be sure that all values of H will be inside it.
Now we shot that every two curves with common endpoints are homotopic with a homotopy that fixes the endpoints. Let $\gamma, \gamma' : [0,1] \to \Omega$ are two curves with common endpoints. We have two homotopies (not fixiing endpoints) H and H' from γ and γ' to the contant curve z_0 respectively. Let the domain of H' be the square in R^2 with corners $(0,0),(1,0),(1,1),(0,1)$ such that $H'(t,1) = z_0$. Let the domain of H be the square $(0,1),(1,1),(1,2),(0,2)$ such that $H(t,1) = z_0$. Since H and H' agree on the intersection we can glue the two domains together and think of a function G defined on the union (the rectangle $(0,0),(1,0),(1,2),0,2)$). We extend G to the triangle $(0,1),(0,0),(-1,1)$ such that it is a constant on each line segment of the form $x + y = c$ for all $c \in [0,1]$. Do the same on the rectangle $(1,1),(2,1),(1,0)$ this time G is a constant on the lines $x - y = c$ for $c \in [0,1]$. In the same way we extend G on the upper triangles $(0,1),(0,2),(-1,1)$ and $(1,2),(1,1),(2,1)$. The domain of G is a hexagon and the value of G on the two left sides is $\gamma(0) = \gamma'(0)$, the value on the two right sides is $\gamma(1) = \gamma'(1)$, the value on the bottom side is γ' and the value on the top is γ. We deform this hexagon to be the unit square to get the desired endpoint fixing homotopy.
(c) Any set that is homeomorphic to a disk is simply connected. That is any set with no "holes" in it - for example the interior of an simple polygon (event if it is not convex or star-shaped).

5 Problem 3.8.22

Consider $g(r) = \displaystyle\int_{|z|=r} f(z)dz$. Cauchy theorem implies that $g(r) = 0$ for all $r < 1$. Now since $f|_{\partial D} = 1/z$ we have $\displaystyle\lim_{r\to 1}\int_{|z|=r} f(z)dz = \int_{|z|=1} \frac{1}{z}dz = \frac{2}{\pi i} \neq 0$. Contradiction.

6 Problem 5.6.3

Let $t = Im(\tau)$. Then we have by the triangle inequality $|\Theta(z|\tau)| \leq \displaystyle\sum_{n=\infty}^{\infty} e^{-\pi n^2 t + 2n|z|}$.
For large enough n, i.e. $n \geq \frac{4|z|}{t}$ we have $\frac{n^2}{2} \geq 2n|z|$ and adding $-n^2 t$ we get $-n^2 t + 2n|z| \leq -n^2 t/2$. Using that to estimate the sum for big $|n|$ we get that $|\Theta(z|t)| \leq \displaystyle\sum_{n\leq -4|z|/t} e^{-\pi n^2 t + 2\pi n|z|} + \sum_{|n|>4|z|/t} e^{-\pi n^2 t + 2\pi n|z|} \leq \sum_{n\leq -4|z|/t} e^{-\pi n^2 t + 2\pi n|z|} + \sum_{|n|>4|z|/t} e^{-n^2 t/2}$.
Every summand in the first sum is bounded by e^{An^2} for some A and since n is bounded by $4|z|/t$ the first sume is bounded by $8|z|/te^{A(4|z|/t)^2} \leq e^{B|z|^2}$ for some B. The second sum is convergent since the sum since $t > 0$ and therefore the function Θ has order 2.

7 Problem 5.6.5

First we show that $-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leq c|z|^{\frac{\alpha}{\alpha-1}}$ for some constant c. This is equivalent to $2\pi|z||t| \leq c|z|^{\frac{\alpha}{\alpha-1}} + \frac{|t|^\alpha}{2}$. If $|z| \leq \frac{|t|^{\alpha-1}}{4\pi}$ then $2\pi|z||t| \leq \frac{|t|^\alpha}{2}$ and we are done. If $|z| > \frac{|t|^{\alpha-1}}{4\pi}$ then for some $c = \frac{2\pi}{(4\pi)^{\frac{1}{\alpha-1}}}$ we have $2\pi|z||t| \leq c|z|^{\frac{\alpha}{\alpha-1}}$ and again we are done.

Usint this inequality we get $|F_\alpha(z)| \leq \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi|z|.|t|}dt = \int_{-\infty}^{\infty} e^{\frac{-|t|^\alpha}{2}} e^{\frac{-|t|^\alpha}{2}+2\pi|z||t|} \leq e^{|z|^{c\alpha/(\alpha-1)}} \int_{-\infty}^{\infty} e^{\frac{-|t|^\alpha}{2}} = de^{|z|^{c\alpha/(\alpha-1)}}$

8 Problem 5.7.1

Fix N and let $D(0, R)$ contains the first N zeroes of f. Let $S_N = \sum_{k=1}^{N}(1 - |z_k|) = \sum_{k=1}^{N} \int_{|z_k|}^{1} 1dr$. Let η_k be the characteristic function of the interval $[|z_k|, 1]$. We have $S_N = \sum_{k=1}^{N} \int_0^1 \eta(r)dr = \int_0^1 (\sum_{k=1}^{N} \eta_k(r))dr \leq \int_0^1 n(r)dr$, where $n(r)$ is the number of zeroes of f at the disk $D(0, r)$. For $r \leq 1$ we have $n(r) \leq \frac{n(r)}{r}$. This means that $S_N \leq \int_0^1 n(r)\frac{dr}{r}$. If $f(0) = 0$ then we have $f(z) = z^m g(z)$ for some integer m and some holomorphic g with $g(0) \neq 0$. The other zeroes of f are precisely the zeroes of g. Thus we have reduced the problem to $f(0) \neq 0$. By the Corollary of the Jensen's equality we get $S_N \leq \int_0^1 n(r)\frac{dr}{r} = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\pi})|d\phi - \log|f(0)| < M$ since f is bounded. The partial sums of the series are boundend and therefore the series converges.

9 Additional problem

Let $g(z) = \frac{z+1}{z-1}$. Fix $z_0 \in \Omega$ and define $f(z) = c + \int_{z_0}^{z} \frac{f'(\xi)}{f(\xi)}d\xi$ where integration is over any curve connecting z_0 to z and lying inside in Ω and c is a constant for which $e^c = g(z_0)$. To show that this is well-defined we need to show that the integral does not depend on the choice of the curve. To show that we need to show that $\sum_\gamma \frac{g'}{g} = 0$ for any closed curve $\gamma \subset \Omega$. By the geometry of Ω it is clear that the point -1 is in the interior of γ if and only if the point 1 is also in the interior of γ. If neither of those points is in the interior then the integral over γ is by the Cauchy theorem. If not calculating the residues we see that $res_{g'/g}(-1) + res_{g'/g}(1) = 0$ and therefore the integral is zero by the residue formula. Now defining f that way we have $f' = g'/g$ and by Cauchy integral

formula we get that $e^g = f$ or $g = \log f$.
To define $f = \sqrt{1-z^2}$ we notice that $\sqrt{1-z^2} = (1-z)\sqrt{\frac{1+z}{1-z}} = (1-z)\sqrt{g(z)}$. Now we define $\sqrt{g} = e^{1/2 \log g}$ which we can do since the $\log g$ is well-defined in Ω.
Locally the antiderivative of $f(z) = \frac{1}{\sqrt{1-z^2}}$ is $\arcsin z = -i \ln(iz + \sqrt{1-z^2})$. Let γ be a cicle that goes exactly one time around the points 1 and -1, i.e. $Ind_1(\gamma) = 1$. Pick any two points $a, b \in \gamma$. We can define two different branches of arcsin on neighborhoods of the two arcs connecting a and b. Call the two branches $\arcsin_1$ and $\arcsin_2$. Then $\int_\gamma f(z)dz = \arcsin_1(a) - \arcsin_1(b) + \arcsin_2(b) - \arcsin_2(a)$. If we define the branches to agree on b then since the difference of the values of log on different branches is $2\pi i$ we have $\arcsin_1(a) - \arcsin_2(a) = -i2\pi i = 2\pi$. Thus if the curve goes around 1 and -1 once the integral is 2π. Now by Cauchy theorem on homotopic curves it follows that $\int_\gamma f(z)dz = 2\pi Ind_1(\gamma)$ as we can homotope a closed curve of winding number n to n closed curves of winding number 1.